

# Quantum Electrodynamics – QED

Starting point: Maxwell's Equations

$$(1) \nabla \cdot \vec{E}(\vec{r}, t) = \frac{1}{\epsilon_0} \rho(\vec{r}, t)$$

$$(2) \nabla \cdot \vec{B}(\vec{r}, t) = 0$$

$$(3) \nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{r}, t)$$

$$(4) \nabla \times \vec{B}(\vec{r}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) + \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{r}, t)$$

Implicit: Charges & Fields in Vacuum

No “medium response”

Same issue as with our introductory example:

Maxwell's eqs are non-local



We need to put the classical description  
in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

$$(1) i\vec{k} \cdot \vec{E}(\vec{k}, t) = \frac{1}{\epsilon_0} G(\vec{k}, t)$$

$$(2) i\vec{k} \cdot \vec{B}(\vec{k}, t) = 0$$

$$(3) i\vec{k} \times \vec{E}(\vec{k}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{k}, t)$$

$$(4) i\vec{k} \times \vec{B}(\vec{k}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{k}, t) + \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{k}, t)$$

Fourier Transform:

$$\left\{ \begin{array}{l} \nabla \cdot \vec{G} \rightsquigarrow i\vec{k} \cdot \vec{h} \\ \nabla \times \vec{G} \rightsquigarrow i\vec{k} \times \vec{h} \end{array} \right.$$

Note: This is a Normal Mode decomposition

No charges -> No coupling between modes  
with different  $\vec{k}$

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## Free Fields - Switch to Fourier Domain

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$$(3) \quad i\vec{k} \times \vec{\mathcal{E}}(\vec{k}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{k}, t)$$

$$(4) \quad i\vec{k} \times \vec{B}(\vec{k}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{\mathcal{E}}(\vec{k}, t) + \frac{1}{\epsilon_0 c^2} \vec{J}(\vec{k}, t)$$

Fourier Transform:  $\left\{ \begin{array}{l} \nabla \cdot \vec{G} \leftrightarrow i\vec{k} \cdot \vec{E} \\ \nabla \times \vec{G} \leftrightarrow i\vec{k} \times \vec{B} \end{array} \right.$

Note: This is a Normal Mode decomposition

No charges  $\rightarrow$  No coupling between modes with different  $\vec{k}$

## Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k}, t) = \vec{\mathcal{E}}_{||}(\vec{k}, t) + \vec{\mathcal{E}}_{\perp}(\vec{k}, t)$$

$$\vec{B}(\vec{k}, t) = \vec{B}_{||}(\vec{k}, t) + \vec{B}_{\perp}(\vec{k}, t) \quad \text{MEq (2)}$$

↑ Entirely Transverse

Note:  $\left\{ \begin{array}{l} \vec{\mathcal{E}}_{||} \text{ is } \frac{\vec{k}}{k} \times \text{the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \\ \vec{\mathcal{E}}_{||} = -\frac{i}{k} i\vec{k} \cdot \vec{\mathcal{E}} \text{ is the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \end{array} \right.$

$$\vec{\mathcal{E}}_{||} = \frac{\vec{k}}{k} \mathcal{E}_{||} = \frac{\vec{k}}{k} \left( -\frac{i}{k} i\vec{k} \cdot \vec{\mathcal{E}} \right) = \frac{\vec{k}}{\epsilon_0 k^2} G(\vec{k}, t)$$

Coulomb field from the charges

Only  $\vec{\mathcal{E}}_{\perp}$  and  $\vec{B}_{\perp}$  are new degrees of freedom beyond the particles  $\rightarrow$  Free Fields

# Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

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MEq (1)

$$\vec{\mathcal{E}}_{||} = \frac{\vec{k}}{k} \mathcal{E}_{||} = \frac{\vec{k}}{k} \left( -i \frac{1}{k} \vec{k} \cdot \vec{\mathcal{E}} \right) = \frac{\vec{k}}{\epsilon_0 k^2} \mathcal{G}(\vec{k}, t)$$

Coulomb field from the charges



Only  $\vec{\mathcal{E}}_{\perp}$  and  $\vec{\mathcal{B}}_{\perp}$  are new degrees of freedom beyond the particles -> Free Fields

Eqs for Transverse Fields, from MEqs (3) & (4)

$$(5a) \frac{\partial}{\partial t} \vec{\mathcal{B}}(\vec{k}, t) = -i \vec{k} \times \vec{\mathcal{E}}_{\perp}(\vec{k}, t)$$

$$(6a) \frac{\partial}{\partial t} \vec{\mathcal{E}}_{\perp}(\vec{k}, t) = c^2 i \vec{k} \times \vec{\mathcal{B}}(\vec{k}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{k}, t)$$

inverse FT

$$(5b) \frac{\partial}{\partial t} \vec{\mathcal{B}}(\vec{r}, t) = -\nabla \times \vec{\mathcal{E}}_{\perp}(\vec{r}, t)$$

$$(6b) \frac{\partial}{\partial t} \vec{\mathcal{E}}_{\perp}(\vec{r}, t) = c^2 \nabla \times \vec{\mathcal{B}}(\vec{r}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{r}, t)$$

combine (5b) & (6b)



Wave Equation for the Free Fields

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{\mathcal{E}}_{\perp}(\vec{r}, t) = \frac{1}{\epsilon_0 c^2} \frac{\partial}{\partial t} \vec{j}_{\perp}(\vec{r}, t)$$

# Quantum Electrodynamics – QED

Eqs for Transverse Fields, from MEqs (3) & (4)

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Wave Equation for the Free Fields

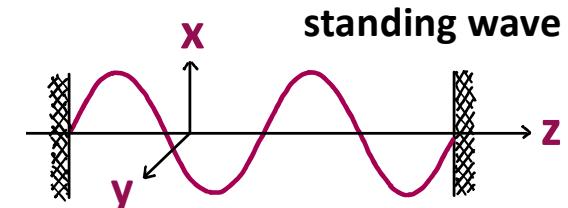
$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_{\perp}(\vec{r}, t) = \frac{1}{\epsilon_0 c^2} \frac{\partial}{\partial t^2} \vec{j}_{\perp}(\vec{r}, t)$$

Normal Modes in a 1D Cavity

Length  $L$

Cross section  $A$

Volume  $V = LA$



Normal Modes are Standing Waves

Let  $\vec{E}(z, t) = \vec{\mathcal{E}}_x E_x(z, t)$  and expand

fiducial mass

$$(7) E_x(z, t) = \sum_j A_j q_j(t) \sin(k_j z), A_j = \sqrt{\frac{2\omega_j m_j}{\epsilon_0 V}}$$

MEq (4) w/no charges

$$\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{r}, t) = \vec{\mathcal{E}}_x \frac{1}{c^2} \sum_j A_j \dot{q}_j(t) \sin(k_j z)$$

$$= \vec{\mathcal{E}}_x \left( \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial z} \right) = -\vec{\mathcal{E}}_x \frac{\partial B_y}{\partial z}$$

$B$  transverse  $\rightarrow B_z = 0$

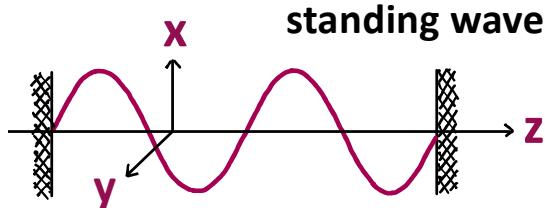
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## Normal Modes in a 1D Cavity

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Let  $\vec{E}(z,t) = \vec{\epsilon}_x E_x(z,t)$  and expand

$$(7) E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), A_j = \sqrt{\frac{2\omega_j m_j}{\epsilon_0 V}}$$

MEq (4) w/no charges

$$\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}_\perp(p,t) = \vec{\epsilon}_x \left[ \frac{1}{c^2} \sum_j A_j \dot{q}_j(t) \sin(k_j z) \right]$$

$$= \vec{\epsilon}_x \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) = -\vec{\epsilon}_x \left[ \frac{\partial B_y}{\partial z} \right]$$

$B$  transverse  $\Rightarrow B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\epsilon}_z \Rightarrow \vec{B}(z,t) = \vec{\epsilon}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = - \sum_j \frac{A_j}{c^2} q_j(t) \sin(k_j z)$$

$$(8) B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\epsilon_0 A}{2} \int_0^L dz (|\vec{E}|^2 + c^2 |\vec{B}|^2) =$$

$$\frac{\epsilon_0 A}{2} \int_0^L dz \sum_j \left[ A_j^2 q_j^2(L)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j^2} \dot{q}_j^2(t)^2 \cos^2(k_j z) \right]$$

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Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting  $A_j^2 = \frac{2\omega_j^2 m_j}{\epsilon_0 V}$  we finally get

$$\mathcal{H} = \sum_j \left[ \frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]$$

Lagrangian for the Classical Field

$$\begin{aligned} \mathcal{L} &= \frac{\epsilon_0 A}{2} \int_0^L dz (C^2 |\vec{B}|^2 - |\vec{E}|^2) \\ &= \sum_j \left[ \frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right] \end{aligned}$$

Check  $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left( \nabla^2 - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_\perp(\vec{r}, t) = 0 \rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

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From Eq. (5a) we see that

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$$\frac{\partial B_y}{\partial z} = - \sum_j \frac{A_j}{c^2} q_j(t) \sin(k_j z)$$



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Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\epsilon_0 A}{2} \int_0^L dz (c^2 |\vec{B}|^2 - |\vec{E}|^2) \quad \checkmark$$

$$= \sum_j \left[ \frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

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# Quantum Electrodynamics – QED

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And Finally:

Conjugate Momentum

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j$$

As before, a collection  
of Harmonic Oscillators,  
ready for quantization!

# Quantum Electrodynamics – QED

To summarize so far...

$$E_x(z, t) = \sum_j A_j q_j(t) \sin(k_j z)$$

$$B_y(z, t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

$$A_j = \sqrt{\frac{\omega_j^2 m_j}{2 \epsilon_0 V}}$$

$$\mathcal{L} = \frac{\epsilon_0 A}{2} \int_0^L dz (c^2 |\vec{B}|^2 - |\vec{E}|^2)$$

$$= \sum_j \left[ \frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j q_j^2 \right]$$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j$$

$$\mathcal{H} = \sum_j \left[ \frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

Classical Fields

Dimensionless Field Variables:

$$Q_j = q_j / q_{0,j}, \quad q_{0,j} = \sqrt{2 \hbar / m_j \omega_j}$$

$$P_j = p_j / p_{0,j}, \quad p_{0,j} = \sqrt{2 \hbar m_j \omega_j}$$



$$\alpha_j(t) = Q_j(t) + i P_j(t) = \alpha_j(0) e^{-i \omega_j t}$$



$$E_x(z, t) = \sum_j A_j q_j(t) \sin(k_j z), \quad \mathcal{E}_j = A_j q_{0,j} = \sqrt{\frac{\hbar \omega_j}{\epsilon_0 V}}$$

$$= \sum_j \mathcal{E}_j [\alpha_j(t) + \alpha_j^*(t)] \sin(k_j z)$$

$$B_y(z, t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

$$= -\frac{i}{c} \sum_j \mathcal{E}_j [\alpha_j(t) - \alpha_j^*(t)] \cos(k_j z)$$

↑  
field  
“per photon”

# Quantum Electrodynamics – QED

## Classical Fields

### Dimensionless Field Variables:

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$$= -\frac{i}{c} \sum_j \mathcal{E}_j [\alpha_j(t) - \alpha_j^*(t)] \cos(k_j z)$$

## Standard Quantization Procedure

$$q_j \rightarrow \hat{q}_j, \quad [q_j, \hat{q}_{j'}] = i\hbar \delta_{jj'}$$

$$\alpha_j(t) \rightarrow \hat{a}_j, \quad [\hat{a}_j, \hat{a}_{j'}^\dagger] = \delta_{jj'}$$

$$\hat{E}_x(z) = \sum_j \mathcal{E}_j (\hat{a}_j + \hat{a}_j^\dagger) \sin(k_j z)$$

$$\hat{B}_y(z) = -\frac{i}{c} \sum_j \mathcal{E}_j (\hat{a}_j - \hat{a}_j^\dagger) \cos(k_j z)$$

## Total Field

$$\hat{\vec{E}}(z) = \vec{\mathcal{E}}_x \hat{E}_x(z) + \vec{\mathcal{E}}_y \hat{E}_y(z)$$

$$\hat{\vec{B}}(z) = \vec{\mathcal{E}}_x \hat{B}_x(z) + \vec{\mathcal{E}}_y \hat{B}_y(z)$$

# Quantum Electrodynamics – QED

## Standard Quantization Procedure

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$$\alpha_j(t) \rightarrow \hat{a}_j, \quad \alpha_j^*(t) \rightarrow \hat{a}_j^*, \quad [\hat{a}_j, \hat{a}_{j'}^+] = \delta_{jj'}$$

$$\hat{E}_x(z) = \sum_j \xi_j (\hat{a}_j + \hat{a}_j^+) \sin(k_j z)$$

$$\hat{B}_y(z) = -\frac{i}{c} \sum_j \xi_j (\hat{a}_j - \hat{a}_j^+) \cos(k_j z)$$

## Total Field

$$\hat{\vec{E}}(z) = \vec{\xi}_x \hat{E}_x(z) + \vec{\xi}_y \hat{E}_y(z)$$

$$\hat{\vec{B}}(z) = \vec{\xi}_x \hat{B}_x(z) + \vec{\xi}_y \hat{B}_y(z)$$

## Note:

These are the Field Operators in the Schrödinger Picture (**t**-dependence in states)

Often advantageous to use Heisenberg Picture  
(**t**-dependence in operators)



$$\alpha_j(t) \rightarrow \hat{a}_j(t) = \hat{a}_j(0) e^{-i\omega_j t}$$

## Field Quantization in Free Space:

Normal Modes :  $\vec{u}_{\vec{k},\lambda}(\vec{r}) = \vec{\xi}_{\vec{k},\lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + C.C.$

$\lambda$ : polarization index

Finite quantization volume:  $\xi_{\vec{k}} = \sqrt{\hbar \omega_{\vec{k}} / 2 \epsilon_0 V}$

**L** large  $\rightarrow$  nature of boundary conditions not important

Periodic boundary conditions

$L \times L \times L$

$$|\vec{k}| = n 2\pi/L$$

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$$\alpha_j(t) \rightarrow \hat{a}_j(t) = \hat{a}_j(0)e^{-i\omega_j t}$$

Field Quantization in Free Space:

Normal Modes :  $\vec{\mu}_{\vec{k},\lambda}(\vec{r}) = \vec{\epsilon}_{\vec{k},\lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + \text{C.C.}$

$\lambda$ : polarization index

Finite quantization volume:  $\epsilon_{\vec{k}} = \sqrt{\hbar \omega_j / 2 \epsilon_0 V}$

$L$  large  $\rightarrow$  nature of boundary conditions not important

Periodic boundary conditions



$$|\vec{k}| = n 2\pi/L$$

Classical Fields (Fourier Sum):

$$\vec{E}_L(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\epsilon}_{\vec{k},\lambda} \epsilon_{\vec{k},\lambda} \alpha_{\vec{k},\lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + \text{C.C.}$$

$$\vec{B}_L(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k},\lambda}}{\hbar c} \epsilon_{\vec{k},\lambda} \alpha_{\vec{k},\lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + \text{C.C.}$$

Quantization:

$$\alpha_{\vec{k},\lambda} \rightarrow \hat{a}_{\vec{k},\lambda}, \quad [\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}^\dagger] = \delta_{\vec{k},\vec{k}'} \delta_{\lambda,\lambda'}$$

$$\alpha_{\vec{k},\lambda}^* \rightarrow \hat{a}_{\vec{k},\lambda}^\dagger, \quad [\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}^\dagger] = [\hat{a}_{\vec{k},\lambda}^\dagger, \hat{a}_{\vec{k}',\lambda'}^\dagger] = 0$$



$$\hat{\vec{E}}_L(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\epsilon}_{\vec{k},\lambda} \epsilon_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} e^{-i(\omega_{\vec{k},\lambda} t - \vec{k} \cdot \vec{r})} + \text{H.C.}$$

$$\hat{\vec{B}}_L(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k},\lambda}}{\hbar c} \epsilon_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} e^{-i(\omega_{\vec{k},\lambda} t - \vec{k} \cdot \vec{r})} + \text{H.C.}$$

– Heisenberg Picture –

# Quantum Electrodynamics – QED

Classical Fields (Fourier Sum):

$$\vec{E}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \mathcal{E}_{\vec{k}, \lambda} \alpha_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + \text{c.c.}$$

$$\vec{B}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k}, \lambda}}{ikc} \mathcal{E}_{\vec{k}, \lambda} \alpha_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + \text{c.c.}$$

Quantization:

$$\alpha_{\vec{k}, \lambda} \rightarrow \hat{a}_{\vec{k}, \lambda}, \quad [\hat{a}_{\vec{k}, \lambda}, \hat{a}_{\vec{k}', \lambda'}^+] = \delta_{\vec{k}, \vec{k}'} \delta_{\lambda, \lambda'}$$

$$\alpha_{\vec{k}, \lambda}^* \rightarrow \hat{a}_{\vec{k}, \lambda}^+, \quad [\hat{a}_{\vec{k}, \lambda}, \hat{a}_{\vec{k}', \lambda'}] = [\hat{a}_{\vec{k}, \lambda}^+, \hat{a}_{\vec{k}', \lambda'}^+] = 0$$



$$\hat{E}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \mathcal{E}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + \text{H.C.}$$

$$\hat{B}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k}, \lambda}}{ikc} \mathcal{E}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + \text{H.C.}$$

– Heisenberg Picture –

Positive & Negative Frequency Components:

$$\hat{E}_L(\vec{r}, t) = \hat{E}^{(+)}(\vec{r}, t) + \hat{E}^{(-)}(\vec{r}, t)$$

$$\hat{E}^{(+)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \mathcal{E}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t + \vec{k} \cdot \vec{r})}$$

$$\hat{E}^{(-)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda}^* \mathcal{E}_{\vec{k}, \lambda}^* \hat{a}_{\vec{k}, \lambda}^+ e^{i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})}$$

Wrap Up:

Read page 13 in handwritten Note Set for brief discussion of different, equivalent ways to put the QED formalism together, e.g.

$$\hat{E}_x \propto (\hat{a}_j + \hat{a}_j^+) \quad \& \quad \hat{B}_y \propto (\hat{a}_j - \hat{a}_j^+)$$

vs

$$\hat{E}_x \propto (\hat{a}_j - \hat{a}_j^+) \quad \& \quad \hat{B}_y \propto (\hat{a}_j + \hat{a}_j^+)$$

# Quantum Electrodynamics – QED

## Positive & Negative Frequency Components:

$$\hat{\vec{E}}_{\perp}(\vec{r}, t) = \hat{\vec{E}}^{(+)}(\vec{r}, t) + \hat{\vec{E}}^{(-)}(\vec{r}, t)$$

$$\hat{\vec{E}}^{(+)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \epsilon_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t + \vec{k} \cdot \vec{r})}$$

$$\hat{\vec{E}}^{(-)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda}^* \epsilon_{\vec{k}, \lambda}^* \hat{a}_{\vec{k}, \lambda}^+ e^{i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})}$$

## Wrap Up:

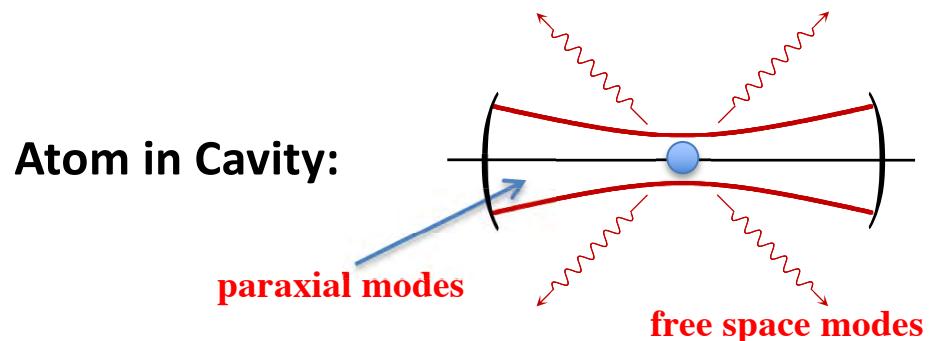
Read page 13 in handwritten Note Set for brief discussion of different, equivalent ways to put the QED formalism together, e. g.

$$\hat{E}_x \propto (\hat{a}_j + \hat{a}_j^+) \quad \& \quad \hat{B}_y \propto (\hat{a}_j - \hat{a}_j^+)$$

VS

$$\hat{E}_x \propto (\hat{a}_j - \hat{a}_j^+) \quad \& \quad \hat{B}_y \propto (\hat{a}_j + \hat{a}_j^+)$$

## Other Normal Modes Sets



Wavepackets: (Milloni & Eberly, Sec. 12.8, p 381) (QED lecture notes, p 16)

Classical field

pulse envelope

$$\vec{E}(\vec{r}, t) = \vec{\epsilon} \epsilon_0 u(z - ct) e^{i(k_0 z - \omega_0 t)} + c.c.$$

Mode volume

$$V = \int d^3r |u(x, y, z - ct)|^2$$

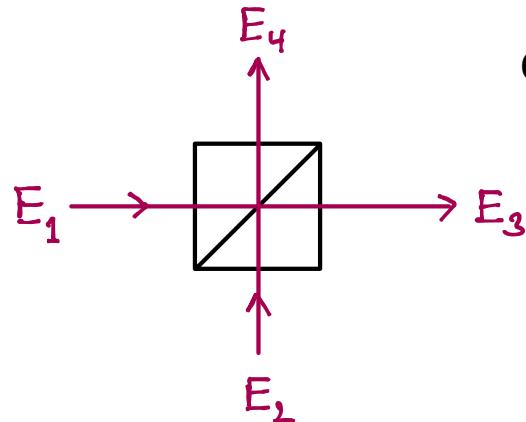
Quantization

$$\epsilon_0 \rightarrow \epsilon_k \alpha_k \rightarrow \epsilon_k \hat{a}_k \quad \text{etc.}$$

Wave-Particle Duality similar for Photons and Phonons

# Application: Classical & Quantum Beamsplitters

## Classical Beamsplitter



Coupled H & V modes

Linear symmetric  
input-output map

$$E_3 = tE_1 + rE_2$$

$$E_4 = rE_1 + tE_2$$

Energy conservation requires

$$|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$$

Choose  $E_1 = 1, E_2 = 0 \rightarrow$

$$|E_3|^2 + |E_4|^2 \propto |t|^2 + |r|^2 = 1$$

Choose  $E_1 = \frac{1}{\sqrt{2}}, E_2 = \frac{i}{\sqrt{2}} \rightarrow$

$$|E_3|^2 + |E_4|^2 \propto \frac{1}{2} |t+r|^2 =$$

$$|t|^2 + |r|^2 + tr^* + rt^* = 1$$

From this it follows that

$$|t|^2 + |r|^2 = 1$$

$$tr^* + rt^* = 0$$

Classical input-output map

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

## Quantum Beamsplitter

Heisenberg  
Picture



Field Operators obey  
Maxwells Eqs

Classical field

$$E_{\perp}(\vec{r}, t) \propto \alpha(t)$$

Quantum equivalent

$$\hat{E}_{\perp}^{(+)}(\vec{r}, t) \propto \hat{\alpha}(t)$$

# Application: Classical & Quantum Beamsplitters

From this it follows that

$$|t|^2 + |r|^2 = 1$$

$$t r^* + r t^* = 0$$

Classical input-output map

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

Quantum Beamsplitter

Heisenberg  
Picture



Field Operators obey  
Maxwells Eqs

Classical field

$$E_{\perp}(\vec{r}, t) \propto \alpha(t)$$

Quantum equivalent

$$\hat{E}_{\perp}^{(+)}(\vec{r}, t) \propto \hat{a}(t)$$

Quantum Beamsplitter

$$\begin{pmatrix} \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}$$



Quantum input-output map

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

Invert Map

$$\hat{a}_3 = t \hat{a}_1 + r \hat{a}_2$$

$$\hat{a}_4 = r \hat{a}_1 + t \hat{a}_2$$

$$\hat{a}_1 = t^* \hat{a}_3 + r^* \hat{a}_4$$

$$\hat{a}_2 = r^* \hat{a}_3 + t^* \hat{a}_4$$

Switch to  
creation  
operators



$$\hat{a}_1^+ = t \hat{a}_3^+ + r \hat{a}_4^+$$

$$\hat{a}_2^+ = r \hat{a}_3^+ + t \hat{a}_4^+$$

# Application: Classical & Quantum Beamsplitters

## Quantum Beamsplitter

$$\begin{pmatrix} \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}$$



## Quantum input-output map

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

## Invert Map

$$\hat{a}_3 = t\hat{a}_1 + r\hat{a}_2$$

$$\hat{a}_4 = r\hat{a}_1 + t\hat{a}_2$$

$$\hat{a}_1 = t^*\hat{a}_3 + r^*\hat{a}_4$$

$$\hat{a}_2 = r^*\hat{a}_3 + t^*\hat{a}_4$$



Switch to  
creation  
operators



$$\begin{aligned} \hat{a}_1^+ &= t\hat{a}_3^+ + r\hat{a}_4^+ \\ \hat{a}_2^+ &= r\hat{a}_3^+ + t\hat{a}_4^+ \end{aligned}$$

## Switch to Schrödinger Picture

General input state:

2-mode vacuum

$$|\Psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (\hat{a}_1^+)^n g_m \frac{1}{\sqrt{m!}} (\hat{a}_2^+)^m |0\rangle$$



The BS maps  $\hat{a}_1^+, \hat{a}_2^+$  to linear combinations of  $\hat{a}_3^+, \hat{a}_4^+$



General output state: (Schrödinger Picture)

$$|\Psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (t\hat{a}_3^+ + r\hat{a}_4^+)^n g_m \frac{1}{\sqrt{m!}} (r\hat{a}_3^+ + t\hat{a}_4^+)^m |0\rangle$$

Example: One-photon input state

$$|\Psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^+ |0\rangle$$

$$|\Psi_{out}\rangle = [t\hat{a}_3^+ + r\hat{a}_4^+] |0\rangle = t|1\rangle_3 |0\rangle_4 + r|0\rangle_3 |1\rangle_4$$

# Application: Classical & Quantum Beamsplitters

Switch to Schrödinger Picture

General input state:

2-mode vacuum

$$|\Psi_{in}\rangle = \sum_{nm} g_n \frac{1}{\sqrt{n!}} (\hat{a}_1^+)^n g_m \frac{1}{\sqrt{m!}} (\hat{a}_2^+)^m |0\rangle$$

The BS maps  $\hat{a}_1^+, \hat{a}_2^+$  to linear combinations of  $\hat{a}_3^+, \hat{a}_4^+$



General output state: (Schrödinger Picture)

$$|\Psi_{out}\rangle = \sum_{nm} g_n \frac{1}{\sqrt{n!}} (t\hat{a}_3^+ + r\hat{a}_4^+)^n g_m \frac{1}{\sqrt{m!}} (r\hat{a}_3^+ + t\hat{a}_4^+)^m |0\rangle$$

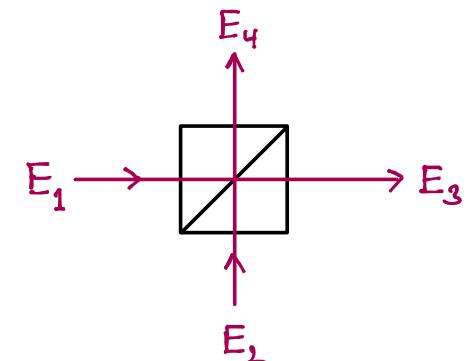
Example: One-photon input state

$$|\Psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^+ |0\rangle$$

$$|\Psi_{out}\rangle = (t\hat{a}_3^+ + r\hat{a}_4^+) |0\rangle = t |1\rangle_3 |0\rangle_4 + r |0\rangle_3 |1\rangle_4$$

50/50 Beamsplitter

$$t = 1/\sqrt{2}, r = i/\sqrt{2}$$



$$|\Psi_{out}\rangle = \frac{1}{\sqrt{2}} (|1\rangle_3 |0\rangle_4 + i |0\rangle_3 |1\rangle_4)$$

Note: This is a Mode Entangled State

- (\*) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

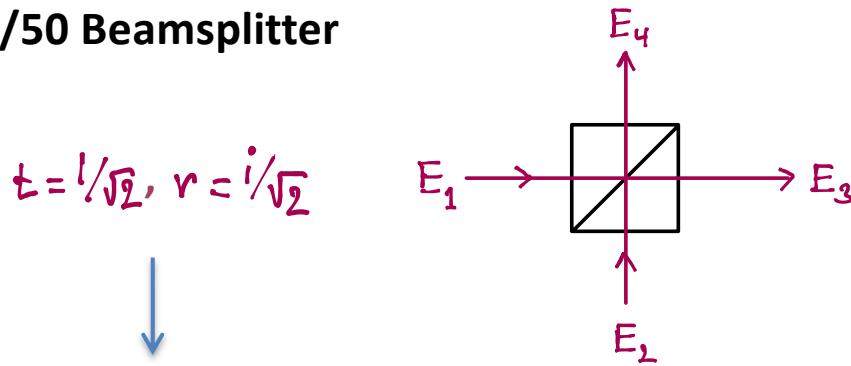
~~$\frac{1}{\sqrt{2}} (|1\rangle_3 + i |0\rangle_3)$~~  to port 3 !

~~$\frac{1}{\sqrt{2}} (|0\rangle_4 + i |1\rangle_4)$~~  to port 4

Viewed on their own, each port is in a mixed state

# Application: Classical & Quantum Beamsplitters

50/50 Beamsplitter



$$|4_{out}\rangle = \frac{1}{\sqrt{2}}(|1\rangle_3|0\rangle_4 + i|0\rangle_3|1\rangle_4)$$

Note: This is a **Mode Entangled State**

- (\*) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

Can we assign states such as, e. g.

~~$\frac{1}{\sqrt{2}}(|1\rangle_3 + i|0\rangle_3)$~~  to port 3 !

~~$\frac{1}{\sqrt{2}}(|0\rangle_4 + |1\rangle_4)$~~  to port 4

Viewed on their own, each port is in a mixed state

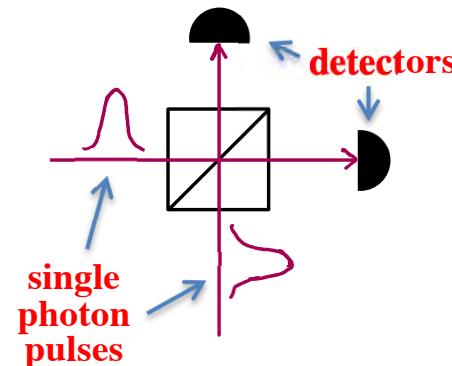
Example: Two-photon input state, 50/50 BS

$$|4_{in}\rangle = \hat{a}_1^+ \hat{a}_2^+ |0\rangle$$

$$\begin{aligned} |4_{out}\rangle &= \frac{1}{2}(\hat{a}_3^+ + i\hat{a}_4^+)(i\hat{a}_3^+ + \hat{a}_4^+) |0\rangle \\ &= \frac{1}{2}(i\hat{a}_3^+ \hat{a}_3^+ + i\hat{a}_4^+ \hat{a}_4^+ + \hat{a}_3^+ \hat{a}_4^+ - \hat{a}_4^+ \hat{a}_3^+) |0\rangle \\ &= \frac{i}{2}(\hat{a}_3^+ \hat{a}_3^+ + \hat{a}_4^+ \hat{a}_4^+) |0\rangle = \frac{i}{\sqrt{2}}(|1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4) \end{aligned}$$

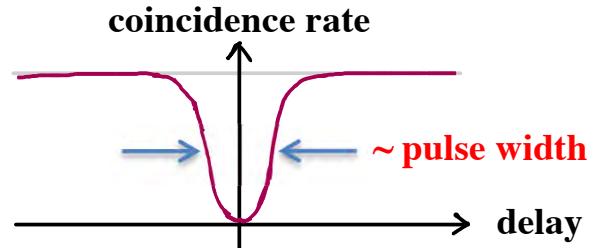
destructive interference

Experiment:



**Coincidence** detections are never seen when pulses overlap  $\rightarrow$  "bunching".

Delay between pulses leads to Coincidence detections.



# Application: Classical & Quantum Beamsplitters

VOLUME 59, NUMBER 18

PHYSICAL REVIEW LETTERS

2 NOVEMBER 1987

## Measurement of Subpicosecond Time Intervals between Two Photons by Interference

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(Received 10 July 1987)

A fourth-order interference technique has been used to measure the time intervals between two photons, and by implication the length of the photon wave packet, produced in the process of parametric down-conversion. The width of the time-interval distribution, which is largely determined by an interference filter, is found to be about 100 fs, with an accuracy that could, in principle, be less than 1 fs.

PACS numbers: 42.50.Bs, 42.65.Re

The usual way to determine the duration of a short pulse of light is to superpose two similar pulses and to measure the overlap with a device having a nonlinear response.<sup>1</sup> The latter might, for example, make use of the process of harmonic generation in a nonlinear medium. Indeed, such a technique was recently used<sup>2</sup> to determine the coherence length of the light generated in the process of parametric down-conversion.<sup>3</sup> The coherence time was found to be of subpicosecond duration, as predicted theoretically.<sup>4</sup> It is, however, in the nature of the technique that it requires very intense light pulses and would be of no use for the measurement of single

phased that the signal and idler photons have no definite phase, and are therefore mutually incoherent, in the sense that they exhibit no second-order interference when brought together at detector D1 or D2. However, fourth-order interference effects occur, as demonstrated by the coincidence counting rate between D1 and D2.<sup>6-8</sup> The experiment has some similarities to another, recently reported, two-photon interference experiment in which fringes were observed and measured, but without the use of a beam splitter.<sup>6</sup>

Although the sum frequency  $\omega_1 + \omega_2$  is very well defined in the experiment, the individual down-shifted frequencies  $\omega_1$  and  $\omega_2$  have larger uncertainties that in gen-

# Application: Classical & Quantum Beamsplitters

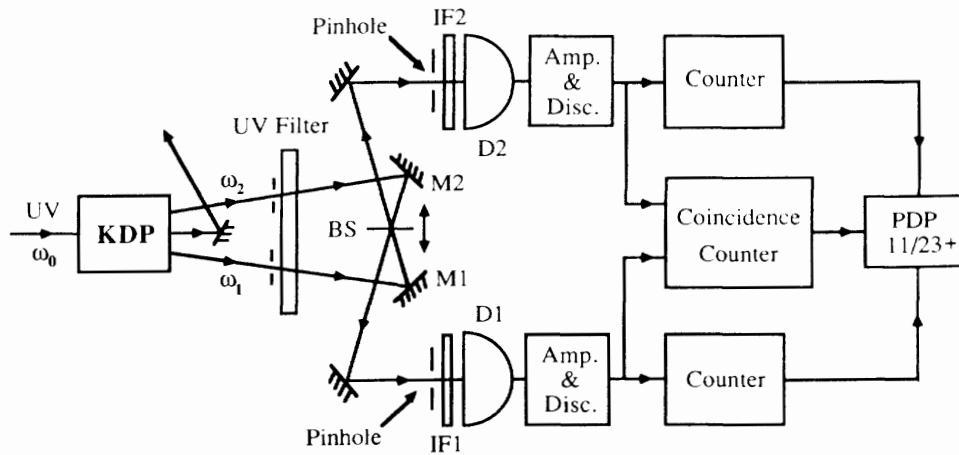


FIG. 1. Outline of the experimental setup.

