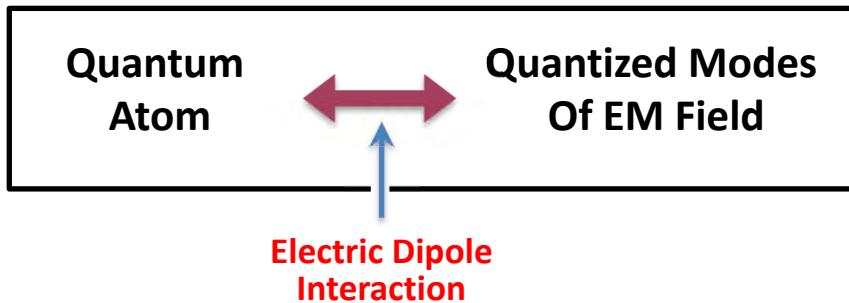


Quantized Light – Matter Interactions

General Problem:



Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \quad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar \omega_{\vec{k}} (\hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{1}{2}) \quad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle \langle i| = \sum_i E_i \hat{\sigma}_i \quad \text{Atom}$$

$$\hat{H}_{AF} = - \vec{p} \cdot \vec{E}(\vec{r}, t) \quad \text{ED interaction}$$

$E_i, |i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$(2) \quad \hat{p} = \sum_{i,j} \vec{p}_{ij} |i\rangle \langle j| = \sum_{i,j} \vec{p}_{ij} \hat{\sigma}_{ij}$$

Field Operator:

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + \text{H.C.}, \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar \omega_{\vec{k}}}{2 \epsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i \vec{k} \cdot \vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$(3) \quad \hat{E}(\vec{r}, t) = \hat{E}(t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^+)$$

Quantized Light – Matter Interactions

Dipole Operator:

(2)

$$\hat{\vec{p}} = \sum_{i,j} \vec{p}_{ij} |; X_j| = \sum_{i,j} \vec{p}_{ij} \hat{\sigma}_{ij}$$

Field Operator:

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \vec{\epsilon}_{\vec{k}}^* \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.C., \quad \vec{\epsilon}_{\vec{k}} = \sqrt{\frac{\hbar \omega_{\vec{k}}}{2 \epsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

- anywhere if $u_{\vec{k}}(\vec{r}) = e^{i \vec{k} \cdot \vec{r}}$ then where
- if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ $\sin(kz) = 1$



(3)

$$\hat{\vec{E}}(\vec{r}, t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\epsilon}_{\vec{k}} \vec{\epsilon}_{\vec{k}}^* (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^*)$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{i,j} \sum_{\vec{k}} - \vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \vec{\epsilon}_{\vec{k}}^* \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^*)$$

$$= \sum_{i,j} \sum_{\vec{k}} \hbar g_{\vec{k}}^{(ij)} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^*)$$

where $g_{\vec{k}}^{(ij)} = - \frac{\vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \vec{\epsilon}_{\vec{k}}^*}{\hbar}$

Rabi Freq., note sign convention

2-level atom $\rightarrow (i, j) = (1, 2) :$

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g_{\vec{k}}^* \hat{\sigma}_{12}) (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^*)$$

Define:

$$\hat{\sigma}_+ = \hat{\sigma}_{21} = [2 \times 1]$$

$$\hat{\sigma}_- = \hat{\sigma}_{12} = [1 \times 2]$$

$$\hat{\sigma}_z = \hat{\sigma}_{22} - \hat{\sigma}_{11} = [2 \times 2] - [1 \times 1]$$

Pauli matrices

$$\hat{\sigma}_x = \frac{1}{2} (\hat{\sigma}_+ + \hat{\sigma}_-)$$

$$\hat{\sigma}_y = \frac{1}{2i} (\hat{\sigma}_+ - \hat{\sigma}_-)$$

$$\hat{\sigma}_z$$

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij} \sum_{\vec{k}} -\vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \epsilon_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^+)$$

$$= \sum_{ij} \sum_{\vec{k}} \hbar g_{\vec{k}}^{(ij)} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^+)$$

where $g_{\vec{k}}^{(ij)} = -\frac{\vec{p}_{ij} \cdot \vec{\epsilon}_{\vec{k}} \epsilon_{\vec{k}}}{\hbar}$

Rabi Freq., note sign convention

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With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}}^* \hat{\sigma}_- \hat{a}_{\vec{k}}^+ + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}} + g_{\vec{k}}^* \hat{\sigma}_+ \hat{a}_{\vec{k}}^+)$$

Energy conservation?



$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}^+)$$

Putting it all together

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \quad (5)$$

$$\sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \sum_{\vec{k}} \hbar (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}}^* \hat{\sigma}_- \hat{a}_{\vec{k}}^+)$$

We changed the zero point for energy by subtracting

$$\sum_{\vec{k}} \frac{1}{2} \hbar \omega_{\vec{k}} \text{ field} \quad \text{and} \quad \frac{1}{2} (E_2 - E_1) \text{ atom}$$

Quantized Light – Matter Interactions

With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_+^* \hat{a}_{\vec{k}}^* + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_-^* \hat{a}_{\vec{k}}^*)$$

Energy conservation?



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Foundational result for
remainder of course

We changed the zero point for energy by subtracting

$$\sum_{\vec{k}} \frac{1}{2} \hbar \omega_{\vec{k}} \quad \text{and} \quad \frac{1}{2} (E_2 - E_1) \quad \begin{array}{c} \overline{\text{field}} \\ \text{atom} \end{array} \quad \begin{array}{c} \overline{\text{---}} \\ \text{---} \end{array} \quad \begin{array}{c} \uparrow \downarrow \\ \frac{1}{2} \hbar \omega_{21} \\ \frac{1}{2} \hbar \omega_{21} \end{array}$$

Quantized Light – Matter Interactions

Begin 04-10-2023

With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}}^+ + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}^+)$$

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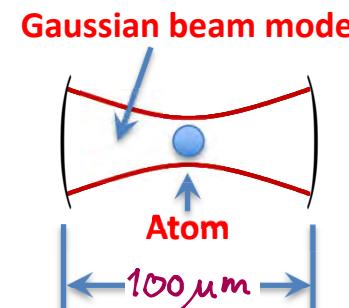
$$\sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{1}{2} \hbar \omega_2 \hat{\sigma}_z + \sum_{\vec{k}} \hbar (g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}^+)$$

We changed the zero point for energy by subtracting

$$\sum_{\vec{k}} \frac{1}{2} \hbar \omega_{\vec{k}} \quad \text{and} \quad \frac{1}{2} (E_2 - E_1) \quad \begin{array}{c} \hline \text{field} \\ \hline \text{atom} \end{array} \quad \begin{array}{c} \uparrow \frac{1}{2} \hbar \omega_2 \\ \downarrow \frac{1}{2} \hbar \omega_2 \end{array}$$

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$$\frac{c}{2L} \gg A_{21}$$

$$|g_{\vec{k}}| \gg A_{21}, \gamma$$

Single-mode (Jaynes-Cummings) Hamiltonian

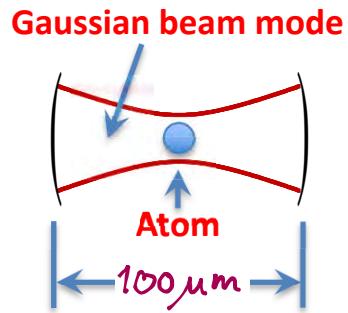
$$\hat{H} = \underbrace{\hbar \omega \hat{a}^+ \hat{a} + \frac{1}{2} \hbar \omega_2 \hat{\sigma}_z}_{H_0} + \underbrace{\hbar g (\hat{\sigma}_+ + \hat{\sigma}_-) (\hat{a}^+ + \hat{a})}_{H_{AF}}$$

End 04-10-2023

Quantized Light – Matter Interactions

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$$\frac{c}{2L} \gg A_{21}$$
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Paradigm for spin-1/2 coupled to QHO

- Atom in high-Q Cavity *)
- Quantum dot in high-Q Cavity
- Rydberg atom in superconducting μw Cavity
- Superconducting qubit in superconducting μw Cavity
- Superconducting qubit in superconducting μw stripline Cavity (circuit QED)
- Trapped ion with quantized COM motion *)

*) Nobel Prize in Physics 2012

Quantized Light – Matter Interactions

Paradigm for spin-1/2 coupled to QHO

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More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega \hat{a}^\dagger \hat{a}}_{H_0} + \frac{1}{2} \hbar\omega_2 \hat{\sigma}_z + \hbar(g_{\vec{k}} \hat{\sigma}_+ + g_{\vec{k}}^* \hat{\sigma}_-) (\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^\dagger)$$

$$H_{AF}$$

$$\text{For simplicity } \vec{n}_{21} = \vec{n}_{12} \rightarrow g_{\vec{k}} = g_{\vec{k}}^* = g_{\vec{k}}$$

Note: \hat{H}_{AF} conserves excitation number, couples $|2, n\rangle \leftrightarrow |1, n+1\rangle$



Series of 2-level systems, one for each n

All 2-level systems are alike
Rabi problem!

Switch to Interaction Picture, p. 6-7 in Notes:

$$\left. \begin{aligned} \hat{H}_s \rightarrow \hat{H}_I &= e^{i\frac{\hat{H}_0}{\hbar}t} \hat{H}_{AF} e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_s(t)\rangle \rightarrow |\psi_I(t)\rangle &= e^{i\frac{\hat{H}_0}{\hbar}t} |\psi_s(t)\rangle \end{aligned} \right\}$$



Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega_2(\hat{\sigma}_z^2)}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_+ + g_{\vec{k}}^*\hat{\sigma}_-)(\hat{a}_{\vec{k}}^\dagger + \hat{a}_{\vec{k}})}_{H_{AF}}$$

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$$\begin{aligned} \hat{H}_{AF} &= \hbar g (\hat{\sigma}_+ \hat{a} e^{i(\omega_2 - \omega)t} + \hat{\sigma}_+ \hat{a}^\dagger e^{i(\omega_2 + \omega)t} \\ &\quad + \hat{\sigma}_- \hat{a} e^{-i(\omega_2 + \omega)t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i(\omega_2 - \omega)t}) \end{aligned}$$

RWA and resonant approximation



Jaynes-Cummings Hamiltonian

$$\hat{H}_I = \hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i\Delta t})$$

$\Delta = \omega_2 - \omega$

Can show $e^{i\omega\hat{a}^\dagger\hat{a}t} \hat{a} e^{-i\omega\hat{a}^\dagger\hat{a}t} = \hat{a} e^{-i\omega t}$
 $e^{i\frac{\omega_2}{2}\hat{\sigma}_z t} \hat{\sigma}_+ e^{-i\frac{\omega_2}{2}\hat{\sigma}_z t} = \hat{\sigma}_+ e^{-i\omega_2 t}$

Quantized Light – Matter Interactions

More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega_2(\hat{\sigma}_z^2)}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_+ + g_{\vec{k}}^*\hat{\sigma}_-)(\hat{a}_{\vec{k}}^\dagger + \hat{a}_{\vec{k}})}_{H_{AF}}$$

For simplicity $\vec{p}_{21} = \vec{p}_{12} \rightarrow g_{\vec{k}} = g_{\vec{k}}^* = g_{\vec{k}}$

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RWA and resonant approximation



Jaynes-Cummings Hamiltonian

$$\hat{H}_I = \hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i\Delta t})$$

$$\Delta = \omega_2 - \omega$$

Eigenstates of $\hat{H}_0 = \hat{H}_F + \hat{H}_A$

State

$$|2,n\rangle$$

$$|1,n+1\rangle$$

Energy

$$\hbar\omega n + \frac{1}{2}\hbar\omega_2$$

$$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_2$$

Quantized Light – Matter Interactions

$$\hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+ \hat{a}^+ e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a} e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a}^+ e^{-i(\omega_{21}-\omega)t})$$

RWA and resonant approximation



Jaynes-Cummings Hamiltonian

$$\hat{H}_I = \hat{H}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^+ e^{-i\Delta t})$$

$\Delta = \omega_{21} - \omega$

Eigenstates of

$$\hat{H}_0 = \hat{H}_F + \hat{H}_A$$

State

Energy

$$|2,n\rangle$$

$$\hbar\omega n + \frac{1}{2}\hbar\omega_{21}$$

$$|1,n+1\rangle$$

$$\hbar\omega(n+1) - \frac{1}{2}\hbar\omega_{21}$$

Cavity QED version of the Rabi Problem

$$|\Psi(0)\rangle = |2,n\rangle$$

$$|\Psi(t)\rangle = C_{1,n+1} |1,n+1\rangle + C_{2,n} |2,n\rangle$$

Matrix elements

$$\langle 2,n | \hat{H}_{AF} | 1,n+1 \rangle = \hbar g \sqrt{n+1} e^{i\Delta t}$$

$$\langle 1,n+1 | \hat{H}_{AF} | 2,n \rangle = \hbar g \sqrt{n+1} e^{-i\Delta t}$$



Schrödinger Equation

$$i\hbar \frac{d}{dt} \begin{pmatrix} C_{1,n+1} \\ C_{2,n} \end{pmatrix} =$$

$$\hbar g \sqrt{n+1} \begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix} \begin{pmatrix} C_{1,n+1} \\ C_{2,n} \end{pmatrix}$$

Quantized Light – Matter Interactions

Cavity QED version of the Rabi Problem

$$|\Psi(0)\rangle = |2,n\rangle$$

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$$\dot{C}_{1,n+1} = -ig\sqrt{n+1} e^{-i\Delta t} C_{2,n}$$

$$\dot{C}_{2,n} = -ig\sqrt{n+1} e^{i\Delta t} C_{1,n+1}$$

Matrix elements

$$\langle 2,n | \hat{H}_{AF} | 1,n+1 \rangle = \hbar g\sqrt{n+1} e^{i\Delta t}$$

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Substitute $C_{1,n+1} \rightarrow C_1$, $C_{2,n} \rightarrow C_2 e^{i\Delta t}$

Looks **exactly** like Semiclassical Rabi problem

Solve for $C_1(0) = 0$, $C_2(0) = 1$



$$C_{2,n}(t) = \left[\cos\left(\frac{\Omega_n t}{2}\right) - i \frac{\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] e^{i\Delta t/2}$$

$$C_{1,n+1} = -i \frac{2g\sqrt{n+1}}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2}$$

$$\Omega_n = (4g^2(n+1) + \Delta^2)^{1/2}$$

Quantized Light – Matter Interactions

$$\dot{c}_{1,n+1} = -ig\sqrt{n+1} e^{-i\Delta t} c_{2,n}$$

$$\dot{c}_{2,n} = -ig\sqrt{n+1} e^{i\Delta t} c_{1,n+1}$$

Substitute $c_{1,n+1} \rightarrow c_1, c_{2,n} \rightarrow c_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $c_1(0) = 0, c_2(0) = 1$



$$c_{2,n}(t) = \left[\cos\left(\frac{\Omega_{nt}}{2}\right) - i \frac{\Delta}{\Omega_n} \sin\left(\frac{\Omega_{nt}}{2}\right) \right] e^{i\Delta t/2}$$

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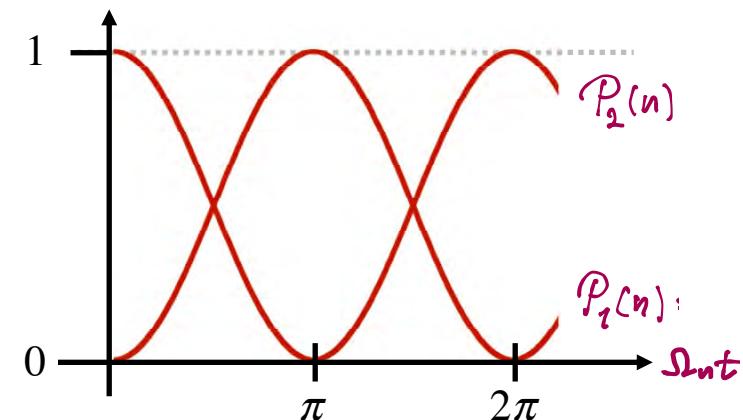
$$\Omega_n = (4g^2(n+1) + \Delta^2)^{1/2}$$

Rabi Oscillations

$$P_2(n) = \cos^2\left(\frac{\Omega_{nt}}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_{nt}}{2}\right)$$

$$P_1(n) = \frac{4g^2(n+1)}{\Omega_n^2} \sin^2\left(\frac{\Omega_{nt}}{2}\right)$$

Example: $\Delta = 0$



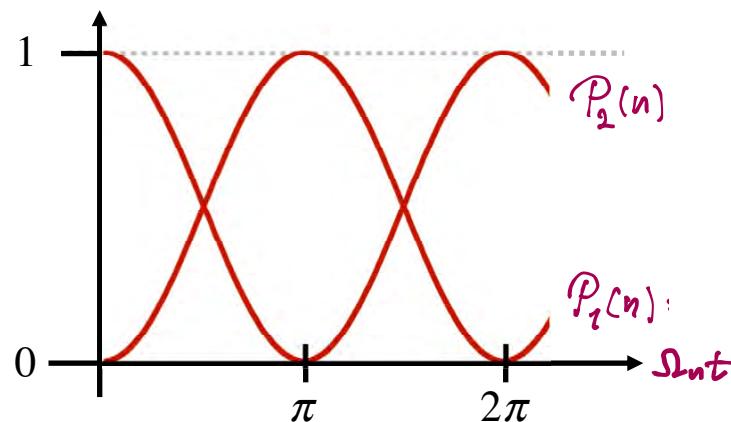
Quantized Light – Matter Interactions

Rabi Oscillations

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Example: $\Delta = 0$



Vacuum Rabi Oscillations

If $|2,0\rangle = |2,0\rangle \rightarrow$ no photons in field

yet $|2,0\rangle$ evolves into $|1,1\rangle$

Uniquely QED phenomenon!

Asymmetry $\begin{cases} |2,n=0\rangle \rightarrow |1,n=1\rangle \\ |1,n=0\rangle \rightarrow |1,n=1\rangle \end{cases}$

holds germ of Spontaneous Decay

Quantized Light – Matter Interactions

Vacuum Rabi Oscillations

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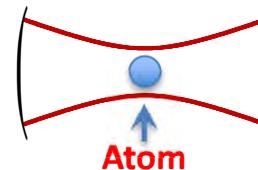
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holds germ of Spontaneous Decay

Next: More Cavity QED



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

$$|\Psi(0)\rangle = |1\rangle \otimes |\alpha\rangle = \sum_n c_n |1,n\rangle, \quad c_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

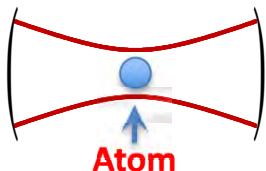
atom field

From Rabi solutions: $\Delta = 0 \rightarrow$

$$c_{1,n} = \cos\left(\frac{\Omega n t}{2}\right), \quad c_{2,n-1} = -i \sin\left(\frac{\Omega n t}{2}\right)$$

Quantized Light – Matter Interactions

Today: More Cavity QED



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

$$|\Psi(0)\rangle = |1\rangle \otimes |\alpha\rangle = \sum_n c_n |1,n\rangle, \quad c_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

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From Rabi solutions: $\Delta=0$ \rightarrow

$$c_{1,n} = \cos\left(\frac{\Omega n t}{2}\right) \quad \rightarrow \quad c_{2,n-1} = -i \sin\left(\frac{\Omega n t}{2}\right)$$

Therefore

uncoupled

$$|\Psi(t)\rangle = c_0 |1,0\rangle + \sum_{n=1}^{\infty} c_n \left[\cos\left(\frac{\Omega n t}{2}\right) |1,n\rangle - i \sin\left(\frac{\Omega n t}{2}\right) |2,n-1\rangle \right]$$

Consider the Atomic Excited State Population

$$\begin{aligned} P_2(t) &= \sum_{n=0}^{\infty} P_{2,n} = \sum_{n=0}^{\infty} |\langle 2,n | \Psi(t) \rangle|^2 \\ &= \sum_{n=0}^{\infty} |c_n|^2 \sin^2\left(\frac{\Omega n t}{2}\right) \\ &= \sum_{n=0}^{\infty} \frac{|\alpha|^2 n!}{n!} e^{-|\alpha|^2} \sin^2\left(\frac{\Omega n t}{2}\right) \end{aligned}$$

Use $|\alpha|^2 = \bar{n}$ and $\Omega_n = 2g\sqrt{n}$

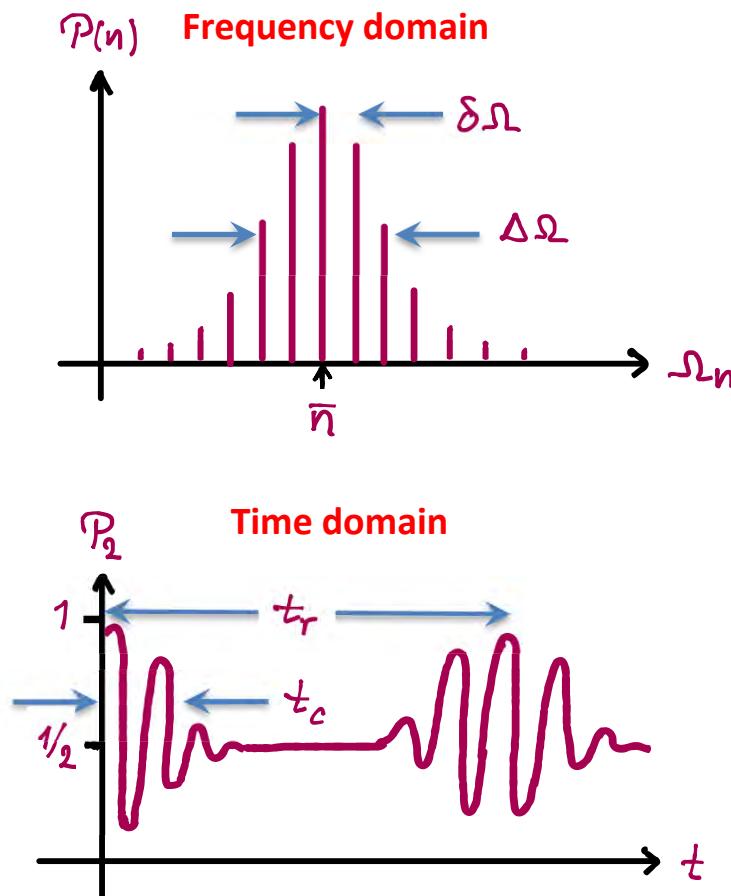
$$P_2(t) = \sum_{n=0}^{\infty} \frac{(\bar{n})^n e^{-\bar{n}}}{n!} \sin^2(g\sqrt{n}t)$$

Quantized Light – Matter Interactions

- Poisson weighted average of sinusoids
- Sinusoids gradually dephase over time



Collapse of oscillation amplitude



Use $\Delta n = \sqrt{\bar{n}} \rightarrow \Delta \Omega \sim \Delta \Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta \Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$t_c = \frac{1}{\Delta \Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}$$

for $\bar{n} \gg \sqrt{\bar{n}}$

Rephasing: when $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})t_r \approx 2\pi m$

Similar arguments \rightarrow Revival time

$$t_r \sim \frac{2\pi}{\Delta \Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}$$

Quantized Light – Matter Interactions

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Collapse & Revival Dynamics

Pure Quantum Phenomenon ("graininess" of photons)

Classical limit

$$\left\{ \begin{array}{l} g \rightarrow 0 \xrightarrow{\Sigma_h \rightarrow 0} t_c \rightarrow \infty \xrightarrow{\bar{n} \rightarrow \infty} \\ \frac{\Delta \Omega}{\Omega_{\bar{n}}} \rightarrow 0 \xrightarrow{\Omega_{\bar{n}} \neq 0} \text{well defined} \end{array} \right.$$

$$\Omega_{\bar{n}} = 2g\sqrt{\bar{n}} = \frac{\vec{P}_{\Omega} \cdot 2\vec{\Sigma}_h \Sigma_h \sqrt{\bar{n}}}{\hbar} = \frac{\vec{P}_{\Omega} \cdot \hat{E}}{\hbar}$$

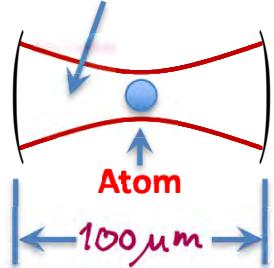
Classical Rabi frequency

mean field $\langle \alpha(t) | \hat{E} | \alpha(t) \rangle$

Quantized Light – Matter Interactions

More Cavity QED – Dressed States

Gaussian beam mode



$$c/\omega_L \gg A_{21}$$

$$|g\vec{E}| \gg A_{21}, \gamma$$

Energy levels of the atom-cavity system

Bare & Dressed States

Return to single - mode result

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

$$\hbar\omega\hat{a}^+\hat{a} + \frac{1}{2}\omega_2\hat{\sigma}_2 \quad \rightarrow H_0$$

$$+ \hbar g(\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^+ e^{-i\Delta t}) \quad \rightarrow H_{AF}$$

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$ 1,n\rangle$	$E_{1,n} = -\frac{\hbar\omega_2}{2} + n\hbar\omega$
$ 2,n-1\rangle$	$E_{2,n-1} = \frac{\hbar\omega_2}{2} + (n-1)\hbar\omega$

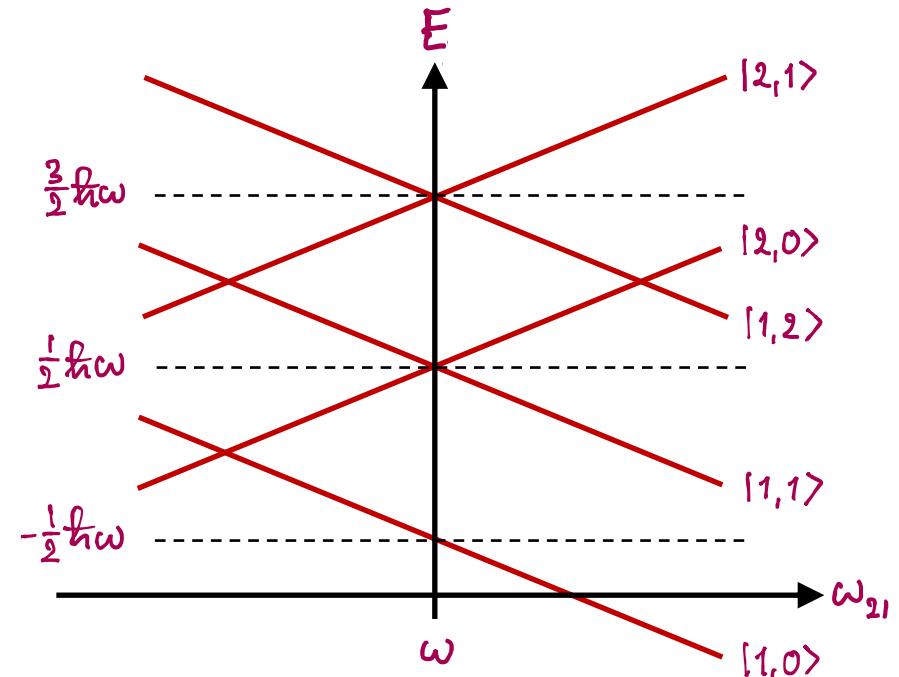
Quantized Light – Matter Interactions

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Imagine we can tune ω_{21}

Energy level diagram



Crossings @ $\omega = \omega_{21}$ are degeneracies of pairs with n shared excitations

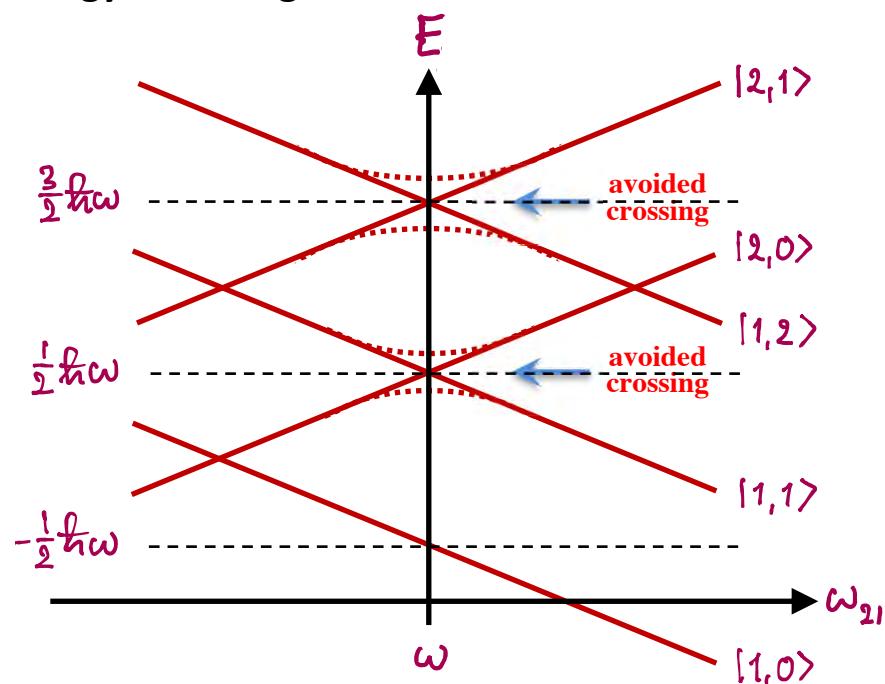
$$\left\{ \begin{array}{ll} n=0 & |1,0\rangle \\ n=1 & \{|1,1\rangle, |2,0\rangle\} \\ n=2 & \{|1,2\rangle, |2,1\rangle\} \\ \vdots & \vdots \\ n & \{|1,n\rangle, |2,n-1\rangle\} \end{array} \right.$$

Quantized Light – Matter Interactions

“Bare” states ($g=0$, eigenstates of \hat{H}_0)

State	Energy
$ 1,n\rangle$	$E_{1,n} = -\frac{\hbar\omega_2}{2} + n\hbar\omega$
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Energy level diagram



“Dressed” states eigenstates of
 $\hat{H} = \hat{H}_0 + \hat{H}_{AF}$

Structure of \hat{H} :

$$\hat{H} = \begin{bmatrix} \hat{H}_0 & & & \\ & \hat{H}_1 & & \\ & & \hat{H}_2 & \\ & \vdots & \ddots & \ddots \end{bmatrix} \quad \begin{array}{l} \text{1x1 block} \\ \text{2x2 blocks} \end{array}$$

Can write this on the form

$$\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n - \frac{1}{2}) \hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}$$

$\Delta = \omega_{21} - \omega$

Quantized Light – Matter Interactions

"Dressed" states

eigenstates of
 $\hat{H} = \hat{H}_0 + \hat{H}_{AF}$

Structure of \hat{H} :

$$\hat{H} = \begin{bmatrix} \hat{H}_0 & & & \\ & \hat{H}_1 & & \\ & & \hat{H}_2 & \\ & & & \ddots \end{bmatrix} \quad \left. \begin{array}{l} 1 \times 1 \text{ block} \\ 2 \times 2 \text{ blocks} \end{array} \right\}$$

Can write this on the form

$$\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(n - \frac{1}{2} \right) \hbar \omega + \begin{bmatrix} -\hbar \Delta / 2 & \hbar g \sqrt{n} \\ \hbar g \sqrt{n} & \hbar \Delta / 2 \end{bmatrix}$$

$\Delta = \omega_{21} - \omega$

Eigenvalues $E_{\pm} = \left(n - \frac{1}{2} \right) \hbar \omega \pm \frac{\hbar}{2} \sqrt{4g^2 n + \Delta^2}$

Eigenstates

$$|+,n\rangle = \frac{\cos(\Theta_n/2)}{\sin(\Theta_n/2)} |1,n\rangle + \frac{\sin(\Theta_n/2)}{\cos(\Theta_n/2)} |2,n-1\rangle$$

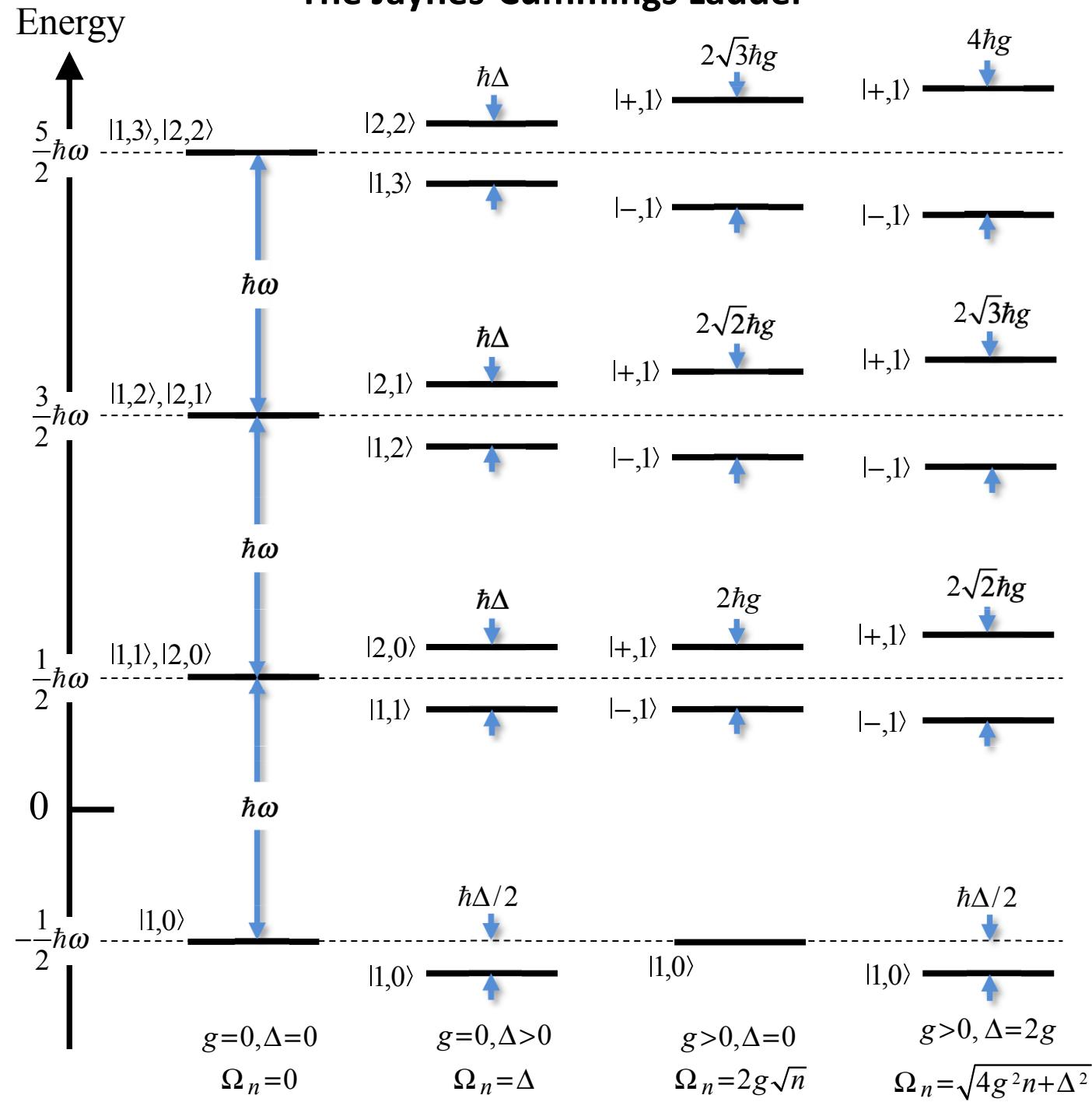
$$|-,n\rangle = -\frac{\sin(\Theta_n/2)}{\cos(\Theta_n/2)} |1,n\rangle + \frac{\cos(\Theta_n/2)}{\sin(\Theta_n/2)} |2,n-1\rangle$$

for $\Delta \leq 0$ $\Delta > 0$

Mixing angle $\tan \Theta_n = -\frac{2g\sqrt{n}}{\Delta}$

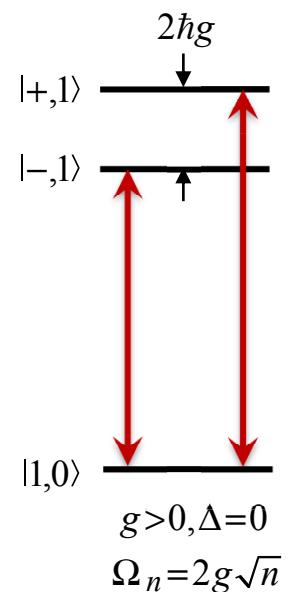
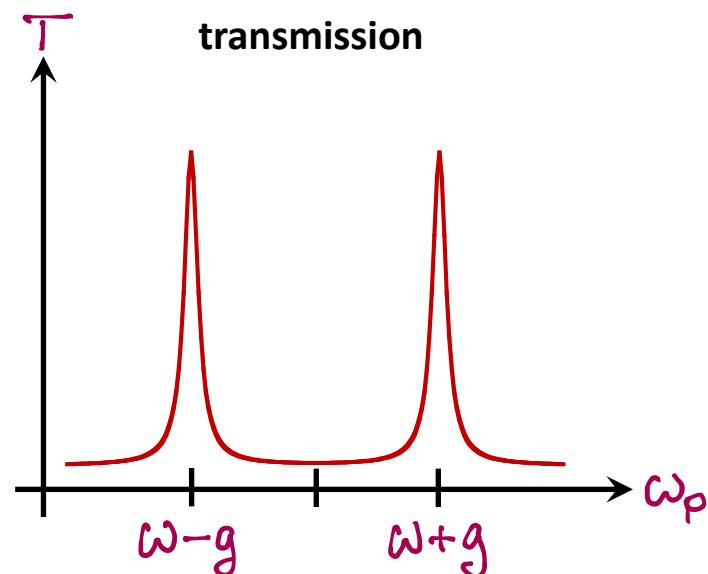
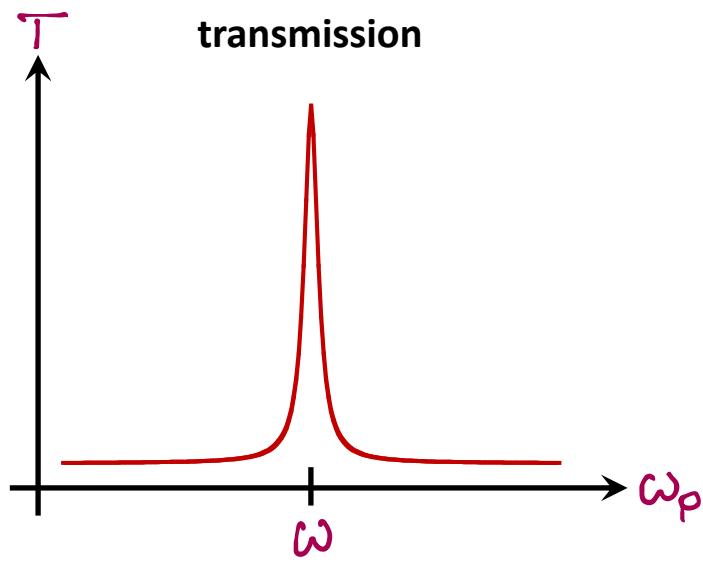
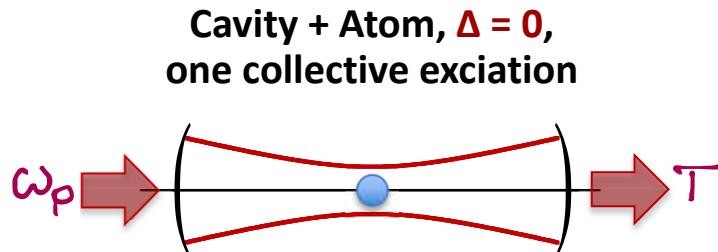
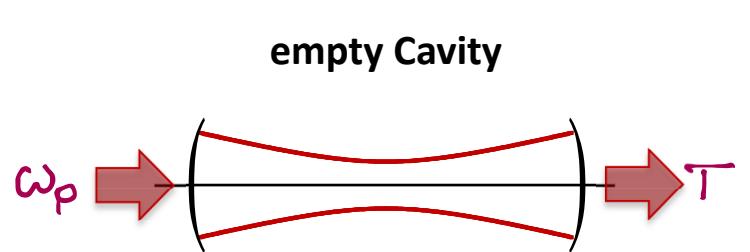
Energy Spectrum?

The Jaynes-Cummings Ladder



Vacuum Rabi splitting

Consider the following experiments



Fluorescence - Mollow Triplet

Coherent state with $\bar{n} = \infty, \frac{\Delta n}{\bar{n}} \rightarrow 0, g \gg 0 \Rightarrow \Omega^2 = 4g^2(\bar{n} + \sqrt{\hbar}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$

