OPTI 544 Solution Set 9, Spring 2023

Problem 1

(a) The vacuum Rabi frequency is $g = -\frac{\vec{p}_{21} \cdot \vec{\varepsilon}_k \vec{\varepsilon}}{\hbar}$, where $\vec{\varepsilon}_k = \sqrt{\frac{\hbar \omega_k}{\varepsilon_0 V}}$.

The field is parallel to the dipole so that $\vec{p}_{21} \cdot \vec{\varepsilon}_k = |\vec{p}_{21}| = p_{21}$, where $A_{21} = \frac{8\pi^2 p_{21}^2}{3\varepsilon_0 \hbar \lambda^3}$.

Putting this together gives us the vacuum Rabi frequency:

$$g = \frac{1}{\hbar} \sqrt{\frac{3\varepsilon_0 \hbar \lambda^3 A_{21}}{8\pi^2}} \sqrt{\frac{\hbar \omega_k}{\varepsilon_0 V}} = \sqrt{\frac{3c\lambda^2 A_{21}}{4\pi V}}$$
$$= \sqrt{\frac{3 \times 3 \times 10^8 \, m \, / \, s \times (852 \times 10^{-9} \, m)^2 \times 3.28 \times 10^7 \, / \, s}{4\pi \times 100 \times 10^{-6} \, m \times \pi (2 \times 10^{-6} \, m)^2}} = 1.165 \times 10^9 \, / \, s$$

We see that g exceeds the rate of energy loss due to decay of the atomic excitation by more than an order of magnitude. This means the system is in a regime where we can see coherent vacuum Rabi oscillations.

(b) The probe transmission is going to show vacuum Rabi splitting. Borrowing a sketch from the class notes, this looks as follows;



Problem II

Problem II

The modified Jaynes-Cummings Hamiltonian is (a)

$$\hat{H} = \underbrace{\hbar\omega \,\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\hbar\omega \left(\hat{\sigma}_{z}^{(1)} + \hat{\sigma}_{z}^{(2)}\right)}_{\hat{H}_{0}} + \underbrace{\hbar g\left(\hat{\sigma}_{+}^{(1)}\hat{a} + \hat{\sigma}_{-}^{(1)}\hat{a}^{\dagger} + \hat{\sigma}_{+}^{(2)}\hat{a} + \hat{\sigma}_{-}^{(2)}\hat{a}^{\dagger}\right)}_{\hat{H}_{AF}}$$
(b) Eigenstates of \hat{H}_{0} : state energy
$$|1,1,0\rangle - \hbar\omega$$

$$|1,1,1\rangle$$

$$|2,1,0\rangle$$

$$|1,2,0\rangle$$

$$0$$

$$(0,0,0,0,0)$$

In basis
$$\{|1,1,0\rangle,|1,1,1\rangle,|2,1,0\rangle,|1,2,0\rangle\}$$
 we have $\hat{H}_{AF} = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & g & g \\ 0 & g & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & g & 0 & 0 \end{pmatrix}$

The upper left 1×1 represents the ground state which is not coupled and has eigenvalue $\lambda = 0$.

The eigenvalues of the 3×3 block are found by setting

$$\begin{vmatrix} -\lambda & g & g \\ g & -\lambda & 0 \\ g & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda g^2 = \lambda (\lambda^2 - 2g^2) = 0 \implies \lambda = 0, \pm \sqrt{2}g$$

Level diagram:

The eigenvalues of \hat{H} are thus



We expect to see vacuum Rabi splitting. Transmission will occur when the probe is resonant with transitions

(d) Restricting to the single-excitation manifold and setting $|\psi_0\rangle = a|1,1,1\rangle + b|2,1,0\rangle + c|1,2,0\rangle$, the relevant eigenvalue problem is

$$\begin{pmatrix} 0 & g & g \\ g & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} gb + gc = 0 \\ ga = 0 \\ |a|^2 + |b|^2 + |c|^2 = 1 \end{cases}$$

From this we see that $|\psi_0\rangle = \frac{1}{\sqrt{2}} (|2,1,0\rangle - |2,1,0\rangle)$ is a solution.

The probe has to add a photon to the cavity and therefore can only couple the ground state $|1,1,0\rangle$ to excited states that have an admixture of $|1,1,1\rangle$, i. e., the transition $|1,1,0\rangle \rightarrow |\psi_0\rangle$ is forbidden and we do not see transmission at $\omega_p = \omega$. The transmission spectrum thus looks like the single atom case, but with a vacuum Rabi splitting that is a factor of $\sqrt{2}$ larger.

<u>Note:</u> It is straighforward to find the other two states in the single-excitation manifold,

$$|\psi_{\pm}\rangle = \frac{1}{2} \left(\pm |1,1,1\rangle + \frac{1}{\sqrt{2}}|2,1,0\rangle + \frac{1}{\sqrt{2}}|1,2,0\rangle\right)$$

These states have a $|1,1,1\rangle$ component and are therefore coupled to the $|1,1,0\rangle$ state by the probe. This confirms that we will still see transmission at $\omega_p = \omega \pm \sqrt{2}g$.

(c)