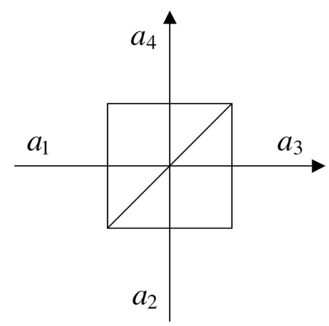
OPTI 544: Homework Set #8
Posted April 14, Due April 24.

I

Consider in the following a 4-port beamsplitter with transmissions and reflection coefficients t and r.



(a) Let the input state be $|\Psi_{in}\rangle = |0\rangle_1 |\alpha\rangle_2$, i. e. the vacuum state in port 1 and a coherent state in port 2. Find the output state, and show that it factorizes into a coherent state in port 3 and a coherent state in port 4.

(b) Show that $\frac{1}{2}\left(\hat{a}e^{-i\varphi} + \hat{a}^{+}e^{i\varphi}\right) = \cos(\varphi)\hat{X} + \sin(\varphi)\hat{Y} = \hat{X}(\varphi)$.

Let the input state be $|\Psi_{in}\rangle = |\psi\rangle_1 |\alpha\rangle_2$, where $\alpha = |\alpha| e^{i\varphi}$ and the quantum state $|\psi\rangle_1$ sent into port 1 is unknown. In *balanced homodyne detection* ($t = 1/\sqrt{2}, r = i/\sqrt{2}$) we measure the difference in optical power in the output ports, integrated over some time τ. This is equivalent to measuring the difference in the number of detected photons, i. e. the observable $\hat{M} = \hat{a}_3^{+}\hat{a}_3 - \hat{a}_4^{+}\hat{a}_4$.

(c) Use the Heisenberg picture, in which the beamsplitter changes the field operators and leaves the state unchanged, $|\Psi_{in}\rangle = |\Psi_{out}\rangle$, to show that

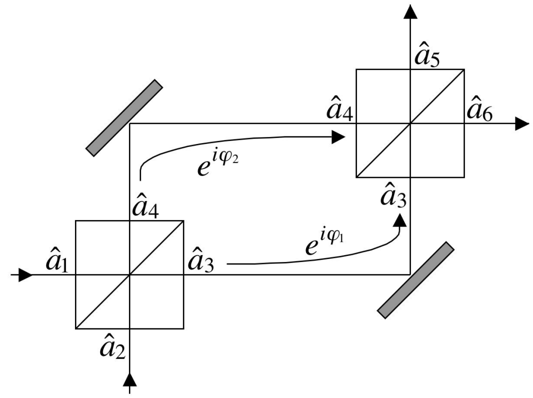
$$\left\langle \hat{M} \right\rangle = 2|\alpha| \left\langle \hat{X}_1(\varphi + \pi/2) \right\rangle, \qquad \Delta M^2 = 4|\alpha|^2 \left[\Delta X_1(\varphi + \pi/2)\right]^2 + \left\langle \hat{N}_1 \right\rangle$$

where $\hat{X}_1$ and $\hat{N}_1$ are the quadrature and photon number operators for the mode entering port 1.

(d) Homodyne detection is often used to detect squeezing. Sketch the variation of ΔM^2 as a function of φ for a quadrature squeezed input state with $\Delta\hat{X} < 1/2$, $\Delta\hat{Y} > 1/2$ and $\Delta X_1 \Delta Y_1 = 1/4$.

II

Consider in the following a Mach-Zender interferometer of the type shown to the left. The input is a product of a coherent state in port one and a vacuum state in port two, $|\Psi_{in}\rangle = |\alpha\rangle_1 |0\rangle_2$. Both beamsplitters are 50/50, and the field picks up unequal phases φ_1 and φ_2 when propagating along the two different paths in the interferometer.



Note: In the Heisenberg picture the phase picked up during propagation changes the operators, $\hat{E}^{(+)} \rightarrow \hat{E}^{(+)} e^{i\varphi}$ and $\hat{a} \rightarrow \hat{a} e^{i\varphi}$. Conversely, in the Schrödinger picture, a coherent state (eigenstate of $\hat{a}$) changes as $|\alpha\rangle \rightarrow |\alpha e^{i\varphi}\rangle$.

(a) Show that the interferometer output is a product of coherent states, $|\Psi_{out}\rangle = |\alpha_5\rangle_5 |\alpha_6\rangle_6$, and derive expressions for the amplitudes α_5, α_6 as functions of α, $\varphi_0 = (\varphi_1 + \varphi_2)/2$ and $\delta\varphi = \varphi_1 - \varphi_2 + \pi/2$.

(b) The power difference in the two output ports integrated over some time is proportional to $\hat{S} = \hat{a}_6^{+}\hat{a}_6 - \hat{a}_5^{+}\hat{a}_5$. Find the expectation value of $\hat{S}$ as a function of α and $\delta\varphi$. Here and in the following use the small angle approximation ($\delta\varphi << 1$).

(c) Find the uncertainty $\Delta\hat{S}^2 = \langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2$ as a function of α and $\delta\varphi$.

(d) The minimum detectable phase difference $\delta\varphi_{\min}$ can be found by setting $\langle \hat{S}(\varphi_{\min}) \rangle = \Delta\hat{S}$. Find $\delta\varphi_{\min}$ as a function of the mean photon number.

III

<u>Note:</u> This problem covers the generation of coherent states based on the radiation from a classical dipole. Completing it is voluntary; you are encouraged to attempt it and should at least check out the 'official" solution set. Whatever you choose to do, the problem will not count towards your homework score.

Consider a single mode of the electromagnetic field, coupled to a classical dipole $d(t)$ located at $z = 0$. As discussed in class, the Hamiltonian for this system can be written as

$$\hat{H}(t) = \hbar\omega(\hat{a}^{+}\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a} + \hat{a}^{+}), \qquad \lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar}$$

In the following, work in the Schrödinger picture.

(a) Calculate the commutators of $\hat{a}$ and $\hat{a}^{+}$ with $\hat{H}(t)$.

(b) Let $\alpha(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle$, where $|\psi(t)\rangle$ is the normalized state vector for the field. Show from the results of (a) that

$$\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) - i\lambda(t)$$

Integrate this differential equation. At time t, what are the mean values of the quadrature operators $\hat{X}(t)$ and $\hat{Y}(t)$?

(c) Let $|\varphi(t)\rangle = [\hat{a} - \alpha(t)]|\psi(t)\rangle$, where $\alpha(t)$ has the value calculated in (b). Using the results from (a) and (b), show that

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle = \left[\hat{H}(t) + \hbar\omega\right]|\varphi(t)\rangle$$

How does the norm of $|\varphi(t)\rangle$ vary with time?

(d) Assuming that $|\psi(0)\rangle$ is an eigenvector of $\hat{a}$ with eigenvalue $\alpha(0)$, show that $|\psi(t)\rangle$ is also an eigenvector of $\hat{a}$, and calculate its eigenvalue.

(e) Assume that at $t = 0$ the field is in the vacuum state. The dipole radiates between times 0 and T and is then removed. When $t > T$, what is the evolution of the mean values $\langle\psi(t)|\hat{X}(t)|\psi(t)\rangle$ and $\langle\psi(t)|\hat{Y}(t)|\psi(t)\rangle$?

(f) Assume that between 0 and T the dipole oscillates so that $\lambda(t) = \lambda_0\cos(\omega' t)$. Derive an expression for $\alpha(T)$ as a function of $\Delta\omega = \omega' - \omega$. Sketch the variation of $|\alpha(T)|^2$ versus $\Delta\omega$. Discuss.

(Adapted from Cohen-Tannoudji's "Quantum Mechanics", Problem 6, p. 636)

A bit of helpful math for Problem III

The linear first-order differential equation

$$\frac{dy}{dx} + p(x)y = q(x)$$

has the solution

$$y(x) = e^{-P(x)}\left[\int_{x_0}^{x} e^{P(x')}q(x')dx' + y(x_0)e^{P(x_0)}\right]$$

where $P(x) = \int p(x)dx$ is the indefinite integral and $y(x_0)$ is the boundary value required to uniquely specify the solution y. The factor $e^{P(x_0)}$ ensures proper behavior for $q(x) = 0$. That this solution is valid is easy to check. For a derivation, see e. g. Arfken, Mathematical Methods for Physicists.

In problem III, substitute $x \to t$, $y(x) \to \alpha(t)$, $p(x) \to i\omega \Rightarrow e^{-P(x)} = e^{-i\omega t}$ and $q(x) \to i\lambda(t)$ to get

$$\alpha(t) = e^{-i\omega t}\left[\int_{t_0}^{t} e^{i\omega t'}i\lambda(t')dt' + \alpha(t_0)e^{i\omega t_0}\right]$$