# OPTI 544 Solution Set 7, Spring 2022

## Problem 1

**(a)** From the Notes/Slides "Quantum Electrodynamics" we have the classical Hamiltonian and Lagrangian,

$$\mathcal{H} = \sum_j \frac{1}{2} m_j \dot{q}_j^2 + \sum_j \frac{1}{2} m_j \omega_j^2 q_j^2$$

$$\mathcal{L} = \sum_j \frac{1}{2} m_j \dot{q}_j^2 - \sum_j \frac{1}{2} m_j \omega_j^2 q_j^2$$

The Lagrange Equation of motion is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \ddot{q}_j + m_j \omega_j^2 q_j = 0 \quad \Rightarrow \quad \ddot{q}_j + \omega_j^2 q_j = 0$$

**(b)** The normal mode expansion of the field is $E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z)$

Substitute in the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \sum_j A_j q_j(t) \sin(k_j z) = \sum_j -A_j k_j^2 q_j(t) \sin(k_j z) - \frac{1}{c^2} \ddot{q}_j \sin(k_j z) = 0$$

This must hold independently for each $j$. Using $k_j = \frac{\omega_j}{c}$ we have

$$\ddot{q}_j + \omega_j^2 q_j = 0$$

This is identical to the result from using the Lagrange formalism in (a) above. Thus, the Lagrangian and the wave equation have the same dynamical eigenmodes governed by the same equations of motion, the two must be equivalent. That in turn tells us that we have found the correct Lagrangian.

## Problem 2

**(a)** We have standing wave modes, and for mode *j* we have

$$\hat{E}_{x,j}(z) = \mathcal{E}_j\left(\hat{a}_j + \hat{a}_j^\dagger\right)\sin(k_j z),$$
$$\hat{B}_{y,j}(z) = \frac{i}{c}\mathcal{E}_j\left(\hat{a}_j - \hat{a}_j^\dagger\right)\cos(k_j z),$$

where $\mathcal{E}_j = \sqrt{\dfrac{\hbar\omega_j}{\varepsilon_0 V}}$ .

**Note:** These are the fields in the Schrödinger picture, i. e., no time dependence in the operators. In this problem we will stick with same-time commutators, but it is straightforward to substitute $\hat{a}_j \to \hat{a}_j \exp(-i\omega_j t)$ to get commutators and uncertainty relations that involve the fields at different times.

**Commutator:** (for simplicity we omit the mode index *j*, observables for different *j* always commute)

$$\begin{aligned}\left[\hat{E}_x(z),\hat{B}_y(z')\right] &= -\frac{i}{c}\mathcal{E}_j^2 \sin(k_j z)\cos(k_j z')\left[\left(\hat{a}_j+\hat{a}_j^\dagger\right)\left(\hat{a}_j-\hat{a}_j^\dagger\right)-\left(\hat{a}_j-\hat{a}_j^\dagger\right)\left(\hat{a}_j+\hat{a}_j^\dagger\right)\right] \\ &= -\frac{i}{c}\mathcal{E}_j^2 \sin(k_j z)\cos(k_j z')\left[\left(\hat{a}_j^2-\hat{a}_j^{\dagger 2}+\hat{a}_j^\dagger\hat{a}_j-\hat{a}_j\hat{a}_j^\dagger\right)-\left(\hat{a}_j^2-\hat{a}_j^{\dagger 2}-\hat{a}_j^\dagger\hat{a}_j+\hat{a}_j\hat{a}_j^\dagger\right)\right] \\ &= -\frac{2i}{c}\mathcal{E}_j^2 \sin(k_j z)\cos(k_j z')\left[\hat{a}_j^\dagger\hat{a}_j-\hat{a}_j\hat{a}_j^\dagger\right] = \frac{2i}{c}\mathcal{E}_j^2 \sin(k_j z)\cos(k_j z')\end{aligned}$$

where in the last step we have used $\hat{a}_j\hat{a}_j^\dagger = \hat{a}_j^\dagger\hat{a}_j + 1$. Thus the field operators are non-commuting except where $z$ and/or $z'$ is at a node for $\hat{E}_x$ and/or $\hat{B}_y$.

**(b)** The uncertainty product is related to the commutator as

$$\boxed{\Delta E_x(z)\Delta B_y(z') \geq \frac{1}{2}\left|\left\langle\left[\hat{E}_x(z),\hat{B}_y(z')\right]\right\rangle\right| = \frac{\hbar\omega_j}{\varepsilon_0 V}\sin(k_j z)\cos(k_j z')}$$

**Note:** The uncertainty product depends on the positions where the fields are measured. Clearly it can be zero if $\sin(k_j z)=0$ and/or $\cos(k_j z')=0$, but that does not pose a contradiction since a measurement at a node of the normal mode standing wave gives no information about the field.

## Problem 3

**(a)** We can write the input as

$$\left|\Psi_{in}\right\rangle = \left(\sqrt{1-\varepsilon}\,\hat{a}_1^\dagger + \sqrt{\varepsilon/2}\,\hat{a}_1^{\dagger 2}\right)\left(\sqrt{1-\varepsilon}\,\hat{a}_2^\dagger + \sqrt{\varepsilon/2}\,\hat{a}_2^{\dagger 2}\right)\left|0\right\rangle$$

Substituting $\hat{a}_1^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_3^\dagger + i\hat{a}_4^\dagger\right)$ and $\hat{a}_2^\dagger = \frac{1}{\sqrt{2}}\left(i\hat{a}_3^\dagger + \hat{a}_4^\dagger\right)$ we get

$$\left|\Psi_{out}\right\rangle = \Big[\frac{i}{2}(1-\varepsilon)\left(i\hat{a}_3^{\dagger 2} + \hat{a}_4^{\dagger 2}\right) - \frac{1-i}{4}\sqrt{\varepsilon(1-\varepsilon)}\left(\hat{a}_3^{\dagger 3} + \hat{a}_4^{\dagger 3} + \hat{a}_3^\dagger \hat{a}_4^{\dagger 2} + \hat{a}_3^{\dagger 2}\hat{a}_4^\dagger\right)$$

$$-\frac{\varepsilon}{8}\left(\hat{a}_3^{\dagger 4} + \hat{a}_4^{\dagger 4}\right) - \frac{\varepsilon}{4}\left(\hat{a}_3^{\dagger 2}\hat{a}_4^{\dagger 3}\right)\Big]\left|0\right\rangle$$

$$= \frac{i}{\sqrt{2}}(1-\varepsilon)\left(\left|2\right\rangle_3\left|0\right\rangle_4 + \left|0\right\rangle_3\left|2\right\rangle_4\right) - \frac{1-i}{4}\sqrt{6\varepsilon(1-\varepsilon)}\left(\left|3\right\rangle_3\left|0\right\rangle_4 + \left|0\right\rangle_3\left|3\right\rangle_4\right)$$

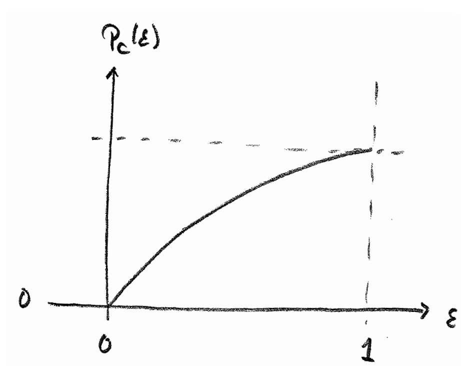
$$-\frac{1-i}{4}\sqrt{2\varepsilon(1-\varepsilon)}\left(\left|1\right\rangle_3\left|2\right\rangle_4 + \left|2\right\rangle_3\left|1\right\rangle_4\right) - \frac{\sqrt{6}\,\varepsilon}{4}\left(\left|4\right\rangle_3\left|0\right\rangle_4 + \left|0\right\rangle_3 4_4\right) - \frac{\varepsilon}{2}\left|2\right\rangle_3\left|2\right\rangle_4$$

**(b)** The output is a superposition of number states with various combinations of photon numbers in each port. The probability of a coincidence detection is the sum of the probability amplitudes squared for all the states that have at least one photon in each port. This gives us

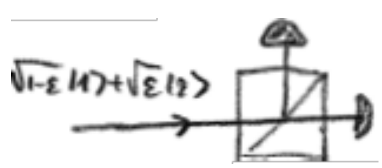
$$P_c = \left(\frac{\varepsilon}{2}\right)^2 + 2\times\left|\frac{1-i}{4}\sqrt{2\varepsilon(1-\varepsilon)}\right|^2 = \frac{1}{4}\varepsilon(2-\varepsilon)$$

$P_c$ takes on a maximum value of $1/4$ at $\varepsilon = 1$. For $\varepsilon \ll 1$ we have $P_c \approx \frac{1}{2}\varepsilon$.

Sketch:




**Note:** To check for two-photon admixture we could just input a single pulse through one port,




Here $P_c \approx \frac{1}{2}\varepsilon$