Problem I

(a) **Motion of a harmonically bound particle.** (Approximately 4 steps, ≤ 100 words total)

(1) Find expressions for the kinetic and potential energy as function of the coordinate q and velocity $\dot{q}$, then write down the Lagrangian for the system.

(2) Check that the Lagrangian yields the familiar classical equation of motion.

(3) From the Lagrangian find the conjugate momentum p.

(4) Replace the classical dynamical variables q and p with Hermitian operators $\hat{q}$ and $\hat{p}$ whose commutator equals $i\hbar$.

(b) **The electromagnetic field in a cavity.** (Approximately 7 steps, ≤ 200 words total)

(1) Identify the free (transverse) fields $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$.

(2) Choose normal modes based on the wave equation and boundary conditions.

(3) Expand $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ in the basis of normal modes. For each mode j, use the coefficients of expansion for $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ as generalized coordinates q_j and generalized velocities $\dot{q}_j$.

(4) For each mode, write the energy of the field in terms of q and $\dot{q}$, to show that the classical Hamiltonian H is the sum of a collection of harmonic oscillators.

(5) Based on H, guess the form of the Lagrangian and verify that it is consistent with the familiar classical equation of motion (the wave equation.)

(6) Use the Lagrangian to find the conjugate momentum p.

(7) Replace the classical variables q and p with Hermitian operators $\hat{q}$ and $\hat{p}$ whose commutator equals $i\hbar$.

Problem II

(a) As shown in class, $|\Psi_{\text{out}}\rangle = t\,|1\rangle_3|0\rangle_4 + r\,|0\rangle_3|1\rangle_4$.

(b) The entangled degrees of freedom are photon number and output port. This is often referred to as photon number - mode entanglement.

(c) The subtlety here is that while the overall (2-port) quantum state is pure, the presence of entanglement means the quantum states in each port viewed separately will be mixed and must be described by a density operator. (see (d) below). There are several ways to approach this. As discussed during lectures, one way is to imagine that a second observer is measuring the photon number in port 4. Due to the correlations built into the entangled state, if the observer detects zero photons in port 4 then there must be one photon in port 3, if one photon in port 4 then zero photons in port 3. Assuming the observer in port 4 does not share information about these outcomes, we know only the probability $|t|^2$ of having zero photons, and the probability $|r|^2$ of having one photon in port 3. In the first case port 3 will contain exactly one photon, while in the second case port 3 will contain exactly zero photons. Thus, there is a $|t|^2$ probability that we have one photon and an $|r|^2$ probability that we have zero photons. Because this is a case of one or the other, we have a mixed state.

(d) The corresponding density operator describing the output in port 3 has the form

$$\hat{\rho}_3 = |t|^2 |1\rangle_3\,{}_3\langle 1| + |r|^2 |0\rangle_3\,{}_3\langle 0|$$

Note that this is generally a mixed state. The exceptions are when

$$|t|^2 = 1\,,\ |r|^2 = 0 \quad \text{or} \quad |t|^2 = 0\,,\ |r|^2 = 1$$

in which cases the state is pure. The degree of mixedness can be found by calculating

$$\text{Tr}\,[\,(\hat{\rho}_3)^2\,] = |t|^4 + |r|^4$$