Problem I - Solution

(a) This is actually our usual 4-level laser system in disguise. Recall that the latter has levels $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$, but that we generally assume from state to the upper lasing level is fast enough that $N_3 = 0$ at all time. Thus we get here the same rate equations as we found for the 4-level system in HW Problem Set 6, Problem I:

(i) $\dot{N}_0 = -PN_0 + \Gamma_{10}N_1 = 0$

(ii) $\dot{N}_1 = -\Gamma_{10}N_1 + \Gamma_{21}N_2 + \sigma\Phi(N_2 - N_1)$

(iii) $\dot{N}_2 = PN_0 - \Gamma_{21}N_2 - \sigma\Phi(N_2 - N_1)$

(b) Below threshold the losses exceed the gain for the light circulating in the laser cavity. As a result the photon flux decays exponentially and the steady state value is $\Phi = 0$.

(c) Combining (a) and (b) we can find the inversion given $\Phi = 0$ and $N_0 + N_1 + N_2 = N$. We already solved that problem in the HW Problem set mentioned above, and found

$$\Delta N = \frac{P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}$$

From this we see that inversion will occur for any for any $P > 0$, if and only if $\Gamma_{10} > \Gamma_{21}$.

Small-signal gain: $g_0 = \sigma\Delta N = \dfrac{\sigma P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}$. Saturated gain: $g = \dfrac{g_0}{1 + \Phi / \Phi_{sat}}$.

Saturation flux: $\Phi_{sat} = \dfrac{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}{(2P + \Gamma_{10})\sigma}$.

(d) Direct pumping from $|0\rangle$ to $|2\rangle$ raises the prospect that the process might be reversible and allow atoms to decay back to $|0\rangle$. This would reduce the effectiveness of pumping.

The parity selection rule precludes that the processes $|0\rangle \rightarrow |2\rangle$, $|2\rangle \rightarrow |1\rangle$, and $|1\rangle \rightarrow |0\rangle$ can be radiative, i. e., driven by absorption or spontaneous emission of light.

Problem II - Solution

(a) We have $\bar{n} = \dfrac{q}{1-q} = 10^{-3} \Rightarrow q = \dfrac{\bar{n}}{1+\bar{n}} \approx 10^{-3} = e^{-\hbar\omega_{21}/k_B T}$. Solving for T gives us

$$T = \hbar\omega_{21} / \ln(10^3) k_B = \frac{1.05\times10^{-34}\,\mathrm{Js}\times 2\pi\times 51\times10^{9}\,\mathrm{s}^{-1}}{6.91\times1.381\times10^{-23}\,\mathrm{J/K}} = \underline{\underline{0.35\mathrm{K}}}$$

(b) This is the QED version of the classic Rabi problem, and the frequency of the Rabi oscillation between the two states is

$$2|g| = \frac{2|\vec{p}_{ij}|}{\hbar}\sqrt{\frac{\hbar\omega_{12}}{2\varepsilon_0 V}} = |\vec{p}_{ij}|\sqrt{\frac{2\omega_{12}}{\hbar\varepsilon_0 (c/\nu)^3}}$$
$$= 1.6\times10^{-29}\,\mathrm{C\,m}\sqrt{\frac{4\pi\times51\times10^{9}\,\mathrm{s}^{-1}}{1.05\times10^{-34}\,\mathrm{Js}\times8.854\times10^{-12}\,\mathrm{F/m}\times[(3\times10^{8}\,\mathrm{m/s})/(51\times10^{9}\,Hz)]^{3}}}$$
$$= 931\,\mathrm{s}^{-1} \approx 148\ \mathrm{Hz}$$

(c) This phenomenon (the fact that the atom evolves out of the excited state even though there are no photons in the field) is referred to as vacuum Rabi oscillation. To make it observable the lifetimes of the atomic states and the photon lifetime in the cavity must all be $\geq (2|g|)^{-1} \sim 20$ ns. So if those lifetimes are too short we might not see it. In this particular version of the experiment we might also be prevented from seeing vacuum Rabi oscillations if the cavity temperature is too high and there are thermal photons present. As for why spontaneous decay is unimportant, the basic intuition is that $\mathrm{A}_{21} \propto \omega^3$, a factor that is reduced by $\sim 10^{13}$ when going from an optical to a microwave frequency. We can be more quantitative by explicitly calculating the spontaneous decay rate for the two-level system,

$$A_{21} = \frac{(2\pi\nu)^3|\vec{p}_{21}|^2}{3\varepsilon_0\hbar\lambda^3} = 3.6\times10^{-5}\ \mathrm{s}^{-1} \quad\Rightarrow\quad \frac{2|g|}{A_{21}} = 2.6\times10^{7} >> 1$$

This shows spontaneous decay is completely negligibe on the timescale of the vacuum Rabi oscillation. The actual experiment was done by Haroche and co-workers many years ago.

Problem III - Solution

(a) First let us find the input-output maps for the creation operators:

$$\hat{a}_H^\dagger \rightarrow \tfrac{1}{\sqrt{2}}\hat{a}_H^\dagger + \tfrac{i}{\sqrt{2}}\hat{a}_V^\dagger$$
$$\hat{a}_V^\dagger \rightarrow \tfrac{i}{\sqrt{2}}\hat{a}_H^\dagger + \tfrac{1}{\sqrt{2}}\hat{a}_V^\dagger$$

From this we immediately get the maps for the basis states

$$|1_H,0_V\rangle = \hat{a}_H^\dagger|0_H,0_V\rangle \rightarrow \left(\tfrac{1}{\sqrt{2}}\hat{a}_H^\dagger + \tfrac{i}{\sqrt{2}}\hat{a}_V^\dagger\right)|0_H,0_V\rangle = \tfrac{1}{\sqrt{2}}(|1_H,0_V\rangle + i|0_H,1_V\rangle)$$
$$|0_H,1_V\rangle = \hat{a}_H^\dagger|0_H,0_V\rangle \rightarrow \left(\tfrac{i}{\sqrt{2}}\hat{a}_H^\dagger + \tfrac{1}{\sqrt{2}}\hat{a}_V^\dagger\right)|0_H,0_V\rangle = \tfrac{1}{\sqrt{2}}(i|1_H,0_V\rangle + |0_H,1_V\rangle)$$

It follows that

$$|\Psi_{\text{in}}^{(+)}\rangle \rightarrow |\Psi_{\text{out}}^{(+)}\rangle = \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{2}}(|1_H,0_V\rangle + i|0_H,1_V\rangle) + i\tfrac{1}{\sqrt{2}}(i|1_H,0_V\rangle + |0_H,1_V\rangle)\right] = i|0_H,1_V\rangle$$
$$|\Psi_{\text{in}}^{(-)}\rangle = \rightarrow |\Psi_{\text{out}}^{(-)}\rangle \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{2}}(|1_H,0_V\rangle + i|0_H,1_V\rangle) - i\tfrac{1}{\sqrt{2}}(i|1_H,0_V\rangle + |0_H,1_V\rangle)\right] = |1_H,0_V\rangle$$

(b) From the sketch on page 4 of the note set "Vector model of the 2-level atom", we get the location of various superpositions when we set $|1\rangle = |1_H,0_V\rangle$ and $|2\rangle = |0_H,1_V\rangle$. This immediately gives us the location of the four states from part (a), see the attached sketch. From this it is clear that the only transformation consistent with the mapping is a rotation by $\pi/4$ around the $\vec{\text{i}}$ axis.

(c) We are not asked to do an actual calculation, but let's do it anyway. Using the input-output map for the basis states, we easily see that

$$|1_H,0_V\rangle\langle 1_H,0_V| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \tfrac{1}{2}(|1_H,0_V\rangle + i|0_H,1_V\rangle)(\langle 1_H,0_V| - i\langle 0_H,1_V|)$$
$$= \tfrac{1}{2}(|1_H,0_V\rangle\langle 1_H,0_V| - i|1_H,0_V\rangle\langle 0_H,1_V| + i|0_H,1_V\rangle\langle 1_H,0_V| + |0_H,1_V\rangle\langle 0_H,1_V|) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$|0_H,1_V\rangle\langle 0_H,1_V| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \tfrac{1}{\sqrt{2}}(i|1_H,0_V\rangle + |0_H,1_V\rangle)\tfrac{1}{\sqrt{2}}(-i\langle 1_H,0_V| + \langle 0_H,1_V|)$$
$$= \tfrac{1}{2}(|1_H,0_V\rangle\langle 1_H,0_V| + i|1_H,0_V\rangle\langle 0_H,1_V| - i|0_H,1_V\rangle\langle 1_H,0_V| + |0_H,1_V\rangle\langle 0_H,1_V|) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

where the matrix representations are in the basis $|1_H,0_V\rangle$, $|0_H,1_V\rangle$. Now we easily see that

$$\rho_{\text{out}} = \frac{p}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{(1-p)}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i(1-2p) \\ -i(1-2p) & 1 \end{pmatrix}$$

For $p = 1/2$ the output state is a maximally mixed state located at the center of the Bloch sphere. For $p = 0$ the input is the pure state at the south pole, and the output is the state $|\Psi_{\text{in}}^{(+)}\rangle$.

The entire calculation above is of course unnecessary: One input is maximally mixed, meaning the output will also be maximally mixed. The other input state is the state $|1_H,0_V\rangle$ at the south pole, which is rotated by $\pi/4$ around the $\vec{\text{i}}$ axis and thus turns into the state $|\Psi_{\text{in}}^{(-)}\rangle$.

(d) With more than one photon in the system there are more than two possible orthogonal states. For example, with two photons we have states $|1,1\rangle$, $|2,0\rangle$, and $|0,2\rangle$. The Bloch sphere can be used only for two-level systems.