Light-Matter Interaction

Hierarchy of Sophistication:

- **Classical**   **Classical light, classical matter**
- **Semiclassical**   **Classical light, quantum matter**
- **Quantum**   **Quantum light, quantum matter**

Possible attitudes:

- **Purist**   **Most complete description possible**
- **Minimalist**   **Quantum only when necessary**
- **Pragmatic**   **Quantum or classical, based on
                   what is simplest and still works**

OPTI 544:   All of the above in turn

Classical Theory of Light-Matter Interaction

Self-consistent, fully classical description

Electromagnetic Field $\longrightarrow$ Atom/Molecule/Solid

Motivation: We will

- Develop <u>Concepts</u> $\alpha(\omega), n, \chi$
- Develop <u>Intuition</u>
- Classical is often adequate, sometimes accurate
- A Quantum Theory has classical limits $\Rightarrow$
  Identify/understand <u>regime of validity</u>
- The Classical description is a useful starting
  point for Nonlinear and Quantum Optics

Light-Matter Interaction

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The Electromagnetic Field: Basic Eqs. in SI Units

Maxwell's eqs.

(no free charges, currents $\Rightarrow$ dielectrics)

(i) $\nabla \cdot \vec{D} = \varrho = 0$ $\vec{D}$: **Dielectric displacement**

(ii) $\nabla \cdot \vec{B} = 0$ $\vec{B}$: **Magnetic induction**

(iii) $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ $\vec{E}$: **Electric field**

(iv) $\nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{J}$ $\vec{H}$: **Magnetic field**

Material Response:

(v) $\vec{B} = \mu_0 \vec{H} + \vec{M}$ $\longleftarrow$ Non-magnetic $\Rightarrow$ $\vec{M} = 0$

(vi) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ $\longleftarrow$ Info about response in dipole moment density (polarization density)

Light-Matter Interaction

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⬅ **Non-magnetic** ➡ $\vec{M} = 0$

⬅ Info about response in dipole moment density (polarization density)

We need equations that describe:

- the behavior of $\vec{E}$ for given $\vec{P}$

- the medium response $\vec{P}$ for given $\vec{E}$

Wave Equation:

Take curl of (iii), then use (iv)

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$

Next, use the identity

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

to obtain $\qquad \vec{D} = \nabla(\nabla \cdot \vec{E}) = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$

Finally, let $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ and use $\varepsilon_0 \mu_0 = \frac{1}{c^2}$

to obtain

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$$-\nabla(\nabla \cdot \vec{E}) + \nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \vec{P}}{\partial t^2}$$

</div>

This is the Wave Equation, still exact in this form

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Transverse Fields

Definition: a field for which $\nabla \cdot \vec{E} \equiv 0$ is <u>Transverse</u>

Example: a plane wave, $\vec{E}(\vec{r},t) = \vec{E}(t)\, e^{i\vec{k}\cdot\vec{r}}$,
where $\vec{E}(t) \perp \vec{k}$, is transverse.
The physical field is $Re[\vec{E}(\vec{r},t)]$

For transverse fields the wave equation simplifies to

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}}$$

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Isotropic Media

Absent a preferred direction, the induced $\vec{P}$ must be parallel to the driving field $\vec{E}$

Linear response, most general case:

$$\vec{D}(t) = \varepsilon_o \vec{E}(t) + \vec{P}(t)$$
$$= \varepsilon_o \vec{E}(t) + \varepsilon_o \int_{-\infty}^{t} dt' R(t-t') \vec{E}(t')$$

where the *response function* $R(t-t')$ is a scalar and we have $R(\tau) = 0$ for $\tau < 0$

Take divergence on both sides and use M.E. (i)

$$\nabla \cdot \vec{D}(t) = \varepsilon_o \nabla \cdot \vec{E}(t) + \varepsilon_o \int_{-\infty}^{t} dt' R(t-t') \nabla \cdot \vec{E}(t') = 0$$

$\Rightarrow$ $\nabla \cdot \vec{E}(t) = -\int_{-\infty}^{t} dt' R(t-t') \nabla \cdot \vec{E}(t')$ for all t

It follows that $\nabla \cdot \vec{E}(t) = 0$ for all t,

<u>OR</u> $R(\tau) = -2\delta(\tau)$

Light-Matter Interaction

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It follows that $\nabla \cdot \vec{E}(t) = 0$ **for all t,**

OR $\quad R(\tau) = -2\delta(\tau)$

Note: if $R(\tau) \propto \delta(\tau)$ (instantaneous response) then

$$\varepsilon_0 \int_{-\infty}^{t} dt' \, R(t-t') \vec{E}(t') = \varepsilon_0 \chi \vec{E}(t)$$

susceptibility

The case $R(\tau) = -2\delta(\tau)$ is an example of negative susceptibility, $\chi < 0$, which only occurs in certain engineered metamaterials.

> **Electric fields are transverse in linear, isotropic dielectric media**
>
> **(including the vacuum)**

Wave Equation in free space

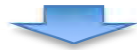$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0$$

Light-Matter Interaction

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Wave Equation in free space

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Monochromatic trial solution $\vec{E}(\vec{r}, t) = \vec{E}_0(\vec{r}) e^{-i\omega t}$

$$\nabla^2 \vec{E}_0(\vec{r}) e^{-i\omega t} + \frac{\omega^2}{c^2} \vec{E}_0(\vec{r}) e^{-i\omega t} = 0$$

Equation for the spatial component alone:

$$\nabla^2 \vec{E}_0(\vec{r}) + |\vec{k}|^2 \vec{E}_0(\vec{r}) = 0 \, , \quad |\vec{k}| = \omega/c$$

Plane wave solutions

$$\vec{E}_0(\vec{r}) = \hat{\varepsilon} E_0 e^{i\vec{k}\cdot\vec{r}} \, , \quad |\vec{k}| = \omega/c$$

Optical Cavities: Here we need to solve the wave equation subject to boundary conditions. See, e. g., Millony & Eberly for examples such as rectangular cavities, Fabryt-Perot etalons, and spherical mirror resonators.
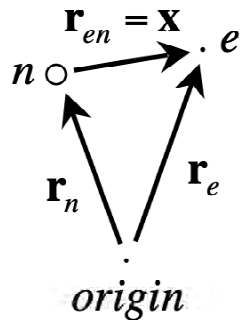
Light-Matter Interaction

Theory of Atomic Response

So far, we have a model for the field. Next, we need a model of how the constituents of the medium responds to the field.

This will allow us to find the polarization density $\vec{P}$ as function of the field $\vec{E}$

Newton:

(i) $\quad m_n \dfrac{d^2}{dt^2}\vec{r}_n(t) = -e\,\vec{E}(\vec{r}_n,t) - \vec{F}_{en}(\vec{r}_{en},t)$

(ii) $\quad m_e \dfrac{d^2}{dt^2}\vec{r}_e(t) = e\,\vec{E}(\vec{r}_e,t) + \vec{F}_{en}(\vec{r}_{en},t)$

This is a standard 2-body problem which we can re-cast as in terms of relative and COM motion.

Classical "atom"

$r_{en} = x$

$n \circ \longrightarrow \cdot\, e$

$r_n \qquad r_e$

$origin$

**Simple model:
nucleus + electron**

Lorentz Force

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

~ 0 if non-relativistic

We define:

$\vec{x} = \vec{r}_{en} = \vec{r}_e - \vec{r}_n$

$\vec{R} = \dfrac{m_e \vec{r}_e + m_n \vec{r}_n}{M}$

$m = \dfrac{m_e m_n}{m_e + m_n} \sim m_e$

$M = m_e + m_n \sim m_n$

$\vec{x}$ Relative coord.

$\vec{R}$ Center-of-mass

m Reduced mass

M Total mass

Light-Matter Interaction

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$$\vec{x} = \vec{r}_{en} = \vec{r}_e - \vec{r}_n \qquad m = \frac{m_e m_n}{m_e + m_n} \sim m_e$$

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$\vec{x}$ Relative coord. m Reduced mass

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Sub into (i), (ii) and rewrite:

$$M\frac{d^2}{dt^2}\vec{R} = e\left[\vec{E}\left(\vec{R}+\frac{m_n}{M}\vec{x},t\right) - \vec{E}\left(\vec{R}-\frac{m_e}{M}\vec{x},t\right)\right]$$

$$m\frac{d^2}{dt^2}\vec{x} = \frac{e}{2}\left[\vec{E}\left(\vec{R}+\frac{m_n}{M}\vec{x},t\right) + \vec{E}\left(\vec{R}-\frac{m_e}{M}\vec{x},t\right)\right]$$
$$+ \vec{F}_{en}(\vec{x}) + \frac{1}{2}(m_n - m_e)\frac{d^2}{dt^2}\vec{R}$$

Basic result, no approximations !

Milloni & Eberly, main text $\Rightarrow$ Set $\vec{R} \approx \vec{r}_n$, $\vec{x} \approx \vec{r}_{en}$
Throw away eq. for $\vec{R}$

Electric Dipole approximation

Atomic dimensions   **Optical Wavelength**

$$|\vec{x}| \sim 1\,\overset{\circ}{A} \qquad \ll \qquad \lambda \sim 10^4\,\overset{\circ}{A}$$

Light-Matter Interaction

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EDA: the field is <u>nearly constant</u> on the scale of an atom

Good approximation: 1st order expansion in $\vec{x}$

$$\vec{E}\left(\vec{R}-\frac{m_e}{M}\vec{x},t\right) \approx \vec{E}(\vec{R},t) - \frac{m_e}{M}(\vec{x}\cdot\nabla)\vec{E}(\vec{R}(t))$$

$$\vec{E}\left(\vec{R}+\frac{m_n}{M}\vec{x},t\right) \approx \vec{E}(\vec{R},t) + \frac{m_n}{M}(\vec{x}\cdot\nabla)\vec{E}(\vec{R}(t))$$

Keeping only terms to leading order in $|\vec{x}|$ **the eqs. of motion then take the simple form**

$$M\frac{d^2}{dt^2}\vec{R} \approx e(\vec{x}\cdot\nabla)\vec{E}(\vec{R},t) \quad \text{COM}$$

$$m\frac{d^2}{dt^2}\vec{x} = e\vec{E}(\vec{R},t) + \vec{F}_{en}(\vec{x}) \quad \text{Rel. Coord.}$$

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Light-Matter Interaction

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Physical Interpretation:

$\vec{p} = e\vec{x}$: <u>electric dipole moment of the atom</u>

If we <u>define</u>:

$$\boxed{V(\vec{x},\vec{R},t) = -\vec{p}\cdot\vec{E}(\vec{R},t)}$$

electric-dipole interaction energy

then the Eqs. of Motion can be recast as

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<u>Note:</u> The COM Eq. is the foundation for a range of laser Atom Traps and Optical Tweezers. We will not explore this further in OPTI 544 lectures, but good review articles can be found in the published literature.

Light-Matter Interaction

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The Electron Oscillator/Lorentz Oscillator

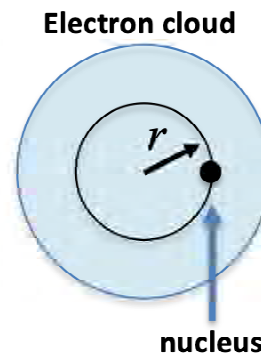**Simple model w/a harmonically bound electron:**
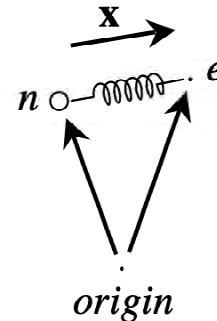
$$\vec{F}_{en} = -m\omega_0^2\vec{x}$$

resonance frequency

This is meant as a model of the atomic **response**, not a model of the atom itself.

Nevertheless: QM suggest the atom consists of a point-like nucleus and a spherical electron cloud

Electron cloud

nucleus

Force from charge inside r as if entire charge was at the center

Force from charge outside r is zero

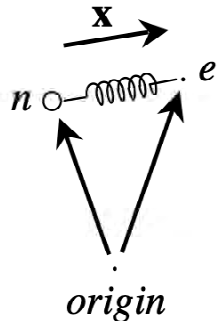Force $F \propto \dfrac{r^3}{r^2} \propto r$

harmonic restoring force

Light-Matter Interaction

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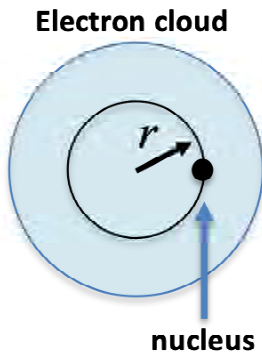
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Now substitute $\vec{F}_{en} = -m\omega_o^2 \vec{x}$ into eq. for $\vec{x}$

$$\frac{d^2}{dt^2}\vec{x} + \omega_o^2 \vec{x} = \frac{e}{m}\vec{E}(\vec{R},t)$$

Combine with $\vec{P} = N\vec{\mu}$, $\vec{\mu} = e\vec{x}$ where N is the number density of atoms. This relates the macroscopic $\vec{P}$ to the microscopic $\vec{x}$

We now have
Maxwell's Equations
The Lorentz model

Maxwell-Lorentz Equations
We can seek self-consistent solutions to wave propagation

Light-Matter Interaction

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Classical Model of Absorption & Dispersion

Maxwell's Eqs: Oscillating dipole loses energy

➡ **Must include damping in Eq, of Motion**

<u>Note</u>: In perfectly homogeneous media the coherently scattered light from a collection of Lorentz oscillators interferes constructively only in the forward direction

λ
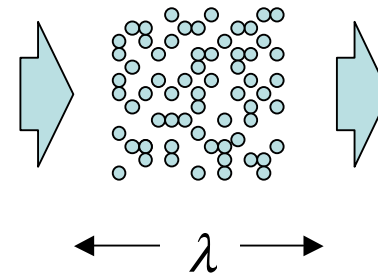
No energy loss for a propagating field
(See note set "Classical Light-Matter")

QM to the rescue: Part of the radiation from quantum mechanical atoms is incoherent.
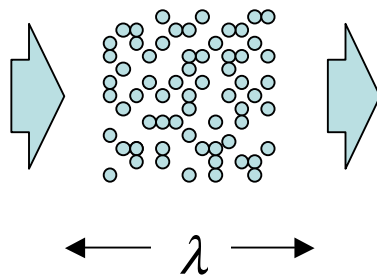
For now we add damping "by hand"

Light-Matter Interaction

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$\longleftarrow \lambda \longrightarrow$

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The Lorentz Model with Damping

We add an ad hoc friction term w/ $\beta \ll \omega_0$

⬆
damping rate

This gives us our basic equation for the atomic response:

$$\frac{d^2}{dt^2}\vec{x} + 2\beta\frac{d}{dt}\vec{x} + \omega_0^2\vec{x} = \frac{e}{m}\vec{E}(\vec{R},t)$$

This type of differential equation generally has both oscillating and decaying terms. Solutions without source terms generally decay as $e^{-\beta t}$

➡ **We adopt a trial solution**

Driving Field	$\vec{E}(\vec{R},t) = \hat{\varepsilon}E_0 e^{-i(\omega t - kz)}$
Response	$\vec{x}(\vec{R},t) = \vec{a} e^{-i(\omega t - kz)}$

⬆
complex amplitude

Light-Matter Interaction

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↑
complex amplitude

Solution for $\vec{a}$:

$$\vec{a} = -\vec{\varepsilon}\,\frac{(e/m)E_0}{\omega^2 - \omega_0^2 + 2i\beta\omega}$$

Physical Quantities:

Field

$$Re[\vec{E}(\vec{R},t)] = \vec{\varepsilon}E_0\cos(\omega t)$$

Dipole ($\vec{\varepsilon}$ real)

$$Re[\vec{p}(\vec{R},t)] = Re[e\vec{x}(\vec{R},t)]$$

$$= \vec{\varepsilon}E_0\frac{e^2}{m}\frac{(\omega_0^2 - \omega^2)\cos(\omega t - kz) + 2\beta\omega\sin(\omega t - kz)}{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2}$$

<u>Note</u>: $\vec{p}$ and $\vec{E}$ generally oscillate out of phase

$\omega \ll \omega_0 \Rightarrow \vec{p}$ & $\vec{E}$ in-phase

$\omega = \omega_0 \Rightarrow \vec{p}$ lags $\vec{E}$ by $\pi/2$

$\omega \gg \omega_0 \Rightarrow \vec{p}$ Lags $\vec{E}$ by π

Best to stick with complex notation !

Light-Matter Interaction

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Video of driven – damped harmonic oscillator

https://www.youtube.com/watch?v=aZNnwQ8HJHU
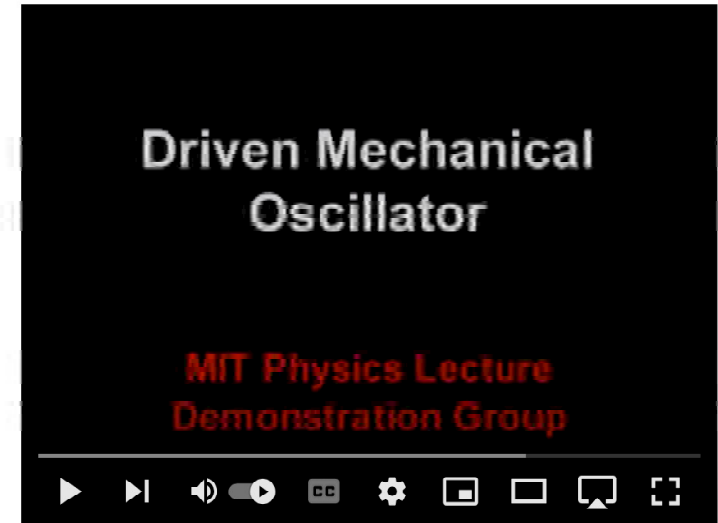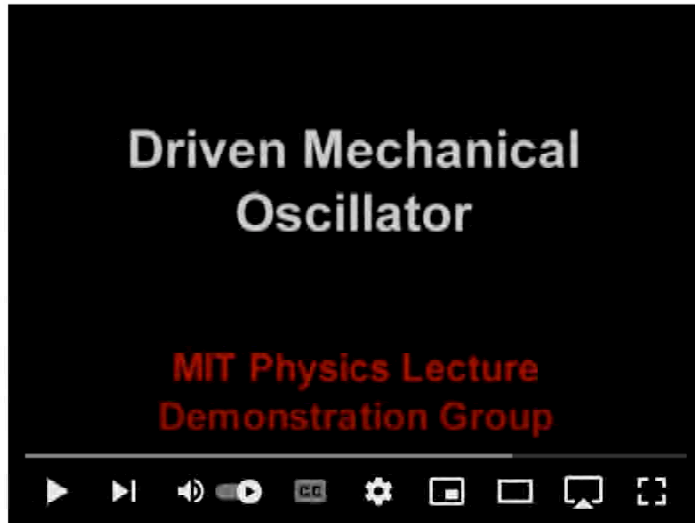

Complex polarizability:

$$\vec{p} = e\vec{x} = e\vec{a}e^{-i(\omega t - kz)} \equiv \alpha(\omega)\vec{\varepsilon}E_0 e^{-i(\omega t - kz)}$$

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} = \frac{e^2}{m} \frac{\omega_0^2 - \omega^2 + 2i\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

Light-Matter Interaction

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Driven Mechanical Oscillator

MIT Physics Lecture Demonstration Group

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Easy to show that if $\vec{E}(\vec{R},t) = \vec{\varepsilon}E_0 e^{-i(\omega t - kz)}$

and $\vec{P} = N\vec{p}$

then the wave equation reduces to

$$\left(-k^2 + \frac{\omega^2}{c^2}\right)\vec{\varepsilon}E_0 e^{-i(\omega t - kz)} =$$

$$-\frac{\omega^2}{c^2}\frac{N\alpha(\omega)}{\varepsilon_0}\vec{\varepsilon}E_0 e^{-i(\omega t - kz)}$$

➡ **plane wave solutions with** $k = n(\omega)\omega/c$ **where**

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left[1 + \frac{N\alpha(\omega)}{\varepsilon_0}\right] \equiv \frac{\omega^2}{c^2}n(\omega)}$$

Complex index of refraction

$$\nabla^2\vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}(\vec{r},t)$$

Light-Matter Interaction

Easy to show that if $\vec{E}(\vec{R},t) = \hat{\varepsilon} E_0 e^{-i(\omega t - k z)}$

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Complex <u>Index of Refraction</u> – Physical discussion

Let $\boxed{n(\omega) = n_R(\omega) + i\, n_i(\omega)}$

Plane wave propagation $k = n(\omega)\frac{\omega}{c}$

$$\vec{E}(z,t) = \hat{\varepsilon} E_0 e^{-i(\omega t - k z)}$$

$$= \hat{\varepsilon} E_0 e^{-i(\omega t - [n(\omega)\omega/c]z)}$$

$$= \hat{\varepsilon} E_0 e^{-n_i(\omega)\omega z/c}\, e^{-i\omega(t - n_R(\omega)z/c)}$$

We can now identify

$$\frac{c}{\omega n_i(\omega)} \longleftarrow \text{attenuation length}$$

$$\frac{c}{n_R(\omega)} \longleftarrow \text{phase velocity}$$

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Light-Matter Interaction

Complex Index of Refraction – Physical discussion

Let
$$n(\omega) = n_R(\omega) + i\, n_i(\omega)$$

Plane wave propagation

$$k = n(\omega)\,\omega/c$$

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Absorption

The intensity of a plane wave field E is

$$I_\omega(z) = \tfrac{1}{2} n_R(\omega)\, c \varepsilon_0\, |\vec{E}(z,t)|^2 = I_0(0)\, e^{-2 n_i(\omega)\omega z/c}$$
$$\equiv I_0\, e^{-a(\omega)z}$$

where the absorption coefficient is

$$a(\omega) \equiv 2 n_I(\omega)\,\omega/c = \frac{2\omega}{c}\, \mathrm{Im}\left[\left(1 + \frac{N\alpha(\omega)}{\varepsilon_0}\right)^{1/2}\right]$$

Possibility of gain ?

No – there is no energy source !

Classical Light-Matter Interaction

Absorption and Dispersion in Gases

Approximations:

$|\omega_0 - \omega| \ll \omega_0, \omega$ near resonance

$|n(\omega)| \sim 1$ weakly polarizable

Let $\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega(\omega_0 - \omega)$

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} = \frac{e^2/2m\omega}{\omega_0 - \omega - i\beta}$$

$$= \frac{e^2}{2m\omega} \frac{\omega_0 - \omega + i\beta}{(\omega_0 - \omega)^2 + \beta^2}$$

Furthermore

$$n(\omega)^2 = 1 + \frac{N\alpha(\omega)}{\varepsilon_0} = 1 + \varepsilon, \ \varepsilon \ll 1$$

Expand to 1st order $(1 + \varepsilon)^{1/2} \approx 1 + \varepsilon/2$

Putting it together

$$n_R(\omega) = 1 + \frac{Ne^2}{4\varepsilon_0 m\omega} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + \beta^2}$$
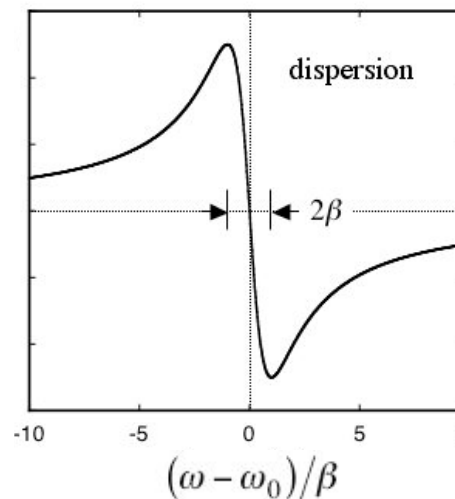
dispersive line shape

$$n_I(\omega) = \frac{Ne^2}{4\varepsilon_0 m\omega} \frac{\beta}{(\omega_0 - \omega)^2 + \beta^2}$$
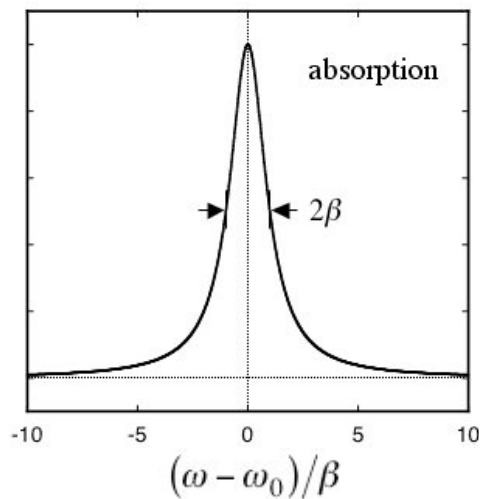
Lorentzian line shape

General behavior:

$n_R(\omega)$ $n_I(\omega)$

$(\omega - \omega_0)/\beta$ $(\omega - \omega_0)/\beta$

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Classical Light-Matter Interaction

Putting it together

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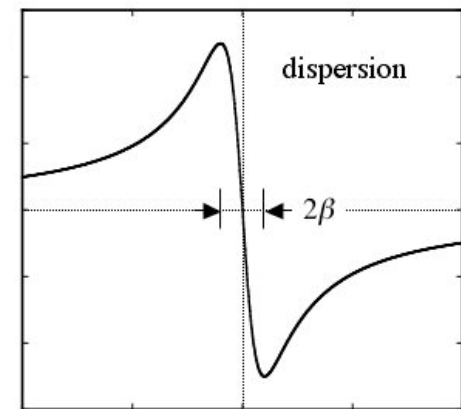
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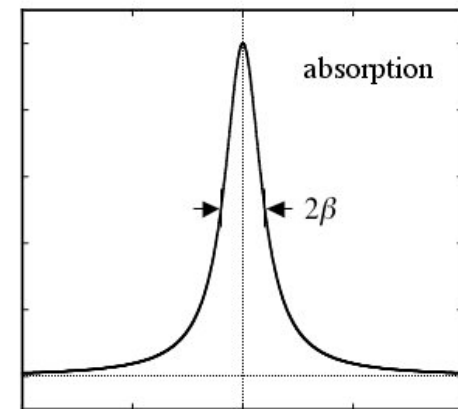
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General behavior:

$n_R(\omega)$ $n_I(\omega)$

Note: for $|\omega_0 - \omega| \gg \beta$

dispersion $\propto \dfrac{1}{(\omega_0 - \omega)}$

absorption $\propto \dfrac{1}{(\omega_0 - \omega)^2}$

➡ **We can have loss-less dispersive media**

Note: If we introduce the detuning $\Delta = (\omega_0 - \omega)$

we can rewrite $n_R(\omega)$, $n_I(\omega)$ as

$$n_R(\Delta) = 1 + \frac{Ne^2}{4\varepsilon_0 m\omega} \frac{\Delta}{\Delta^2 + \beta^2}$$

$$n_I(\Delta) = \frac{Ne^2}{4\varepsilon_0 m\omega} \frac{\beta}{\Delta^2 + \beta^2}$$

From the above we see that

$$n_R(\omega) < 1 \text{ for } \omega > \omega_0 \quad \Rightarrow \quad \frac{c}{n_R(\omega)} \gg c$$

Superluminal propagation?

Classical Light-Matter Interaction

Note:

$$\text{dispersion} \propto \frac{1}{(\omega_0 - \omega)}$$

$$\text{absorption} \propto \frac{1}{(\omega_0 - \omega)^2}$$

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Superluminal propagation ?

Free Electrons

Consider the limit $\omega \gg \omega_0$

➡ **effectively unbound electrons**

This is a reasonable model of plasmas & metals

In this limit we have

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} \approx -\frac{e^2}{m\omega} \quad ➡$$

$$n(\omega) = \sqrt{1 + \frac{N(\alpha)}{\varepsilon_0}} \approx \sqrt{1 - \frac{Ne^2}{\varepsilon_0 m\omega^2}} \equiv \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

We introduce the Plasma Frequency :

$$\omega_p = \sqrt{\frac{Ne^2}{\varepsilon_0 m}}$$

Classical Light-Matter Interaction

Free Electrons

Consider the limit $\omega \gg \omega_0$

➡️ effectively unbound electrons

This is a reasonable model of plasmas & metals

In this limit we have

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} \simeq -\frac{e^2}{m\omega} \quad \Rightarrow$$

$$n(\omega) = \sqrt{1 + \frac{N(\alpha)}{\varepsilon_0}} \approx \sqrt{1 - \frac{Ne^2}{\varepsilon_0 m\omega^2}} \equiv \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

We introduce the **Plasma Frequency** :

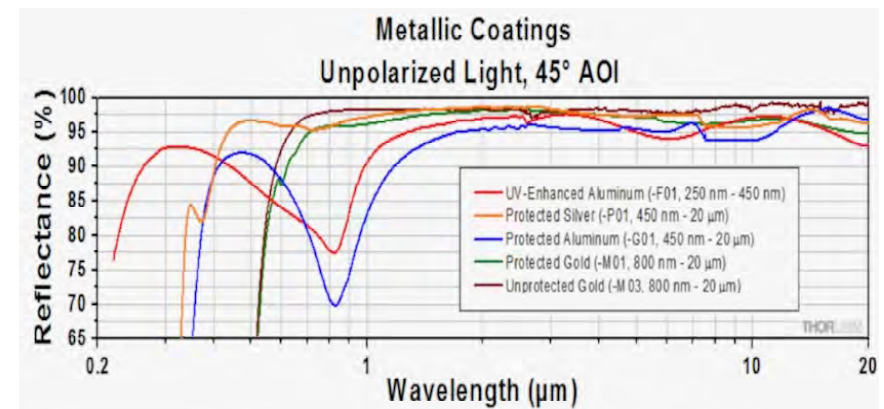$$\omega_p = \sqrt{\frac{Ne^2}{\varepsilon_0 m}}$$

Let
$$\left.\begin{array}{c} \omega_0 \ll \omega \ll \omega_p \\[4pt] |\omega_0 - \omega| \gg \beta \end{array}\right\} \Rightarrow$$
$n(\omega)$ **purely imaginary**
- but no loss !

We now have

$$\vec{E}(z,t) = \hat{\varepsilon} E_0 e^{-i\omega[t - n(\omega)z/c]}$$

$$= \hat{\varepsilon} E_0 e^{-i\omega t} e^{i(z/c)\sqrt{\omega^2 - \omega_p^2}}$$

$$= \hat{\varepsilon} E_0 e^{-i\omega t} e^{-b(\omega)z}$$

where $\quad b(\omega) = -\frac{i}{c}\sqrt{\omega^2 - \omega_p^2}$

Reflection at surface, penetration depth $\quad \sim 1/b(\omega)$

Metallic Coatings — Unpolarized Light, 45° AOI

Examples of this kind of medium includes plasmas, and metals such as aluminum, silver and gold which are known to be excellent mirrors for visible and IR radiation.