OPTI 544: Homework Set #5
Distributed March 16, Due March 27.

I

An ensemble of two-level atoms are driven by a resonant (classical) light field. While the detuning $\Delta = 0$ is uniform across the ensemble, the resonant Rabi frequency χ varies from atom to atom.

(a) Consider two atoms starting in the ground state at $t = 0$, and subsequently evolving with Rabi frequencies $(3/4)\chi_0$ and $(5/4)\chi_0$, where χ_0 is real and positive. Sketch the trajectories of the two atoms on separate Bloch spheres for $t = 0$ to $t = 2\pi / \chi_0$, being careful to indicate the Bloch vectors at $t = 0, t = \pi / \chi_0$, and $t = 2\pi / \chi_0$.

(b) Next, consider what happens if, at $t = \pi / \chi_0$, we change the phase of the driving field by 180° and then continue until $t = 2\pi / \chi_0$. Sketch the trajectories of the two atoms on separate Bloch spheres, being careful to indicate the Bloch vectors at $t = 0, t = \pi / \chi_0$, and $t = 2\pi / \chi_0$.

The pulse sequence in (b) is an example of a so-called *rotary spin echo*.

II

Consider a plane-wave monochromatic electric field, $\vec{E}(\vec{r},t) = \vec{\varepsilon}\mathcal{E}(z,t)e^{-i(\omega t - kz)}$ in complex notation. To obtain self-consistent solutions to the wave equation we need the complex polarization density $\vec{P}(z,t)$ induced by this field. In a semiclassical description $\vec{P}(z,t)$ is obtained from the quantum mechanical expectation value $\langle \hat{\vec{p}} \rangle$ for the individual atoms. Note that $\langle \hat{\vec{p}} \rangle$ is *real-valued*, and that we must extract the part proportional to $\vec{\varepsilon}e^{-i\omega t}$ for use in the complex wave equation. Thus we set $\vec{P}(z,t) = \vec{\varepsilon}\, 2N\mu^* \rho_{21}(z,t)e^{-i(\omega t - kz)}$, where $\mu^* = \vec{p}_{12} \cdot \vec{\varepsilon}^*$.

(a) Apply the slowly varying envelope approximation to $\mathcal{E}(z,t)$ and $\rho_{21}(z,t)$, and derive the following equation for the field envelope:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = \frac{ik}{\varepsilon_0} N\mu^* \rho_{21}(z,t)$$

(b) Show that in steady state

$$\frac{\partial}{\partial z}\mathcal{E}(z,t) = \frac{1}{2}(a - i\delta)(\rho_{22} - \rho_{11})\mathcal{E}(z,t)$$

and derive expressions for the attenuation/gain coefficient a and dispersion δ.

III

In the spirit of Problem II above, we will study the steady state dipole moment $\langle \hat{\vec{p}} \rangle$ of a 2-level atom interacting with a driving field $\vec{E}(t) = \vec{\varepsilon}\mathcal{E}_0 e^{-i\omega t}$. The interaction is characterized by a Rabi frequency $\chi = \vec{p}_{21} \cdot \vec{\varepsilon}\mathcal{E}_0 / \hbar$, a detuning $\Delta = \omega_{21} - \omega$, and an excited state lifetime A_{21}.

(a) Write down an expression for the dipole moment $\langle \hat{\vec{p}} \rangle$ in steady state. Note: There is only the one driving field here so use steady state values for the coherences and populations as found in HW Set 3 Problem III.

(b) Write down an expression for the complex polarizability $\alpha(\omega)$, defined as for the Lorentz atom ($\vec{p}(t) = \alpha(\omega)\vec{E}(t)$). As in Problem II, use the part of $\langle \hat{\vec{p}} \rangle$ that is parallel to $\vec{E}(t)$ and has the appropriate complex time dependence.

<u>Hint</u>: Your expression for $\alpha(\omega)$ should be proportional to $\dfrac{\Delta + i\beta}{\Delta^2 + \tilde{\beta}^2}$, where $\tilde{\beta}$ is the power broadened linewidth.

(c) Using the result in (b), find the real and imaginary indices of refraction near resonance, in the limit where $n(\omega) = 1 + \varepsilon$, $|\varepsilon| << 1$. Write your expressions for $n_R(\omega)$ and $n_I(\omega)$ in terms of I/I_{SAT} rather than the Rabi frequency. Sketch the behavior of both, in the limit $I << I_{SAT}$.

(d) Compare your result in (c) to the predictions of the classical Lorentz model. Then sketch the quantum result when $I >> I_{SAT}$. Discuss the effects of saturation.