OPTI 544 Solution Set 3, Spring 2021

Problem 1

(a) The equations of motion for the probability amplitudes in the RWA, using the "slow" variables and setting $\Delta = \delta = 0$, are

$$\boxed{\dot{b}_1 = -i\frac{\chi_1}{2}b_2, \qquad \dot{b}_2 = -i\left(\frac{\chi_1}{2}b_1 + \frac{\chi_2}{2}b_3\right), \qquad \dot{b}_3 = -i\frac{\chi_2}{2}b_2.}$$

(b) If $b_2(0) = 0$ then $b_2(t) = 0$ if and only if $\dot{b}_2 = 0$ at all times. That requires

$$\left(\frac{\chi_1}{2}b_1 + \frac{\chi_2}{2}b_3\right) = 0 \;\Rightarrow\; b_3 = -\frac{\chi_1}{\chi_2}b_1.$$

Now $$|b_1|^2 + |b_3|^2 = 1 \;\Rightarrow\; |b_1|^2 + \frac{\chi_1^2}{\chi_2^2}|b_1|^2 = 1 \;\Rightarrow\; |b_1|^2 = \frac{\chi_1^2}{\chi_1^2 + \chi_2^2}.$$

Consistent with the above, we choose $$\boxed{b_1 = \frac{\chi_2}{\sqrt{\chi_1^2 + \chi_2^2}}, \qquad b_3 = \frac{-\chi_1}{\sqrt{\chi_1^2 + \chi_2^2}}}$$

Note: If $b_2(t) = 0$ then $\dot{b}_1 = \dot{b}_3 \Rightarrow b_1,\ b_3$ are constant

(c) If the driving field does not lead to a non-zero probability amplitude in the excited state $|2\rangle$, then there can be no induced dipole moment . This is because $|1\rangle$ and $|3\rangle$ of necessity must be of the same parity to allow for Raman coupling in the first place. That means there can be no absorption or emission of light, and as a result the wave propagates without loss of intensity.

Note: The state found in (b) above is referred to as a "dark state".

Problem 2

(a) Taking the outer product of the state vectors we find

$$\rho_1 = \frac{1}{3}\begin{pmatrix} 1 \\ i\sqrt{2} \\ 0 \end{pmatrix}\left(1, -i\sqrt{2}, 0\right) = \frac{1}{3}\begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{Check: } \mathrm{Tr}\left[\rho_1\right] = 1$$

$$\rho_2 = \frac{1}{5}\begin{pmatrix} 1+i \\ 1 \\ -i\sqrt{2} \end{pmatrix}\left(1-i,\ i\sqrt{2}\right) = \frac{1}{5}\begin{pmatrix} 2 & 1+i & -(1-i) \\ 1-i & 1 & i\sqrt{2} \\ -(1+i) & -i\sqrt{2} & 2 \end{pmatrix}. \quad \text{Check: } \mathrm{Tr}\left[\rho_2\right] = 1$$

Then $$\rho = \frac{1}{2}\left(\rho_1 + \rho_2\right) = \frac{1}{2}\times\frac{1}{15}\left[\begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1+i & -(1-i) \\ 1-i & 1 & i\sqrt{2} \\ -(1+i) & -i\sqrt{2} & 2 \end{pmatrix}\right]$$

$$= \frac{1}{30}\begin{pmatrix} 11 & 3+i(3-5\sqrt{2}) & -(1-i)3\sqrt{2} \\ 3-i(3-5\sqrt{2}) & 13 & i\,3\sqrt{2} \\ -(1+i)\,3\sqrt{2} & -i\,3\sqrt{2} & 6 \end{pmatrix}. \quad \text{Check: } \mathrm{Tr}\left[\rho\right] = 1$$

(b) To check for purity we can compute ρ^2 and then check if $\rho^2 \neq \rho$ or $\mathrm{Tr}[\rho^2] < 1$, either of which would tell us that the state is mixed. For example, in this case it is straightforward though somewhat tedious to show that $\mathrm{Tr}[\rho^2] = (19 - 2\sqrt{2})/30 = 0.539 < 1$, which tells us the state is mixed.

There is an easier way if we know the ensemble decomposition, $\rho = \Sigma_k p_k |\psi_k\rangle\langle\psi_k|$. Namely, that ρ is mixed if one or more of the $|\psi_k\rangle$ are linearly independent of the other. In our case we can confirm this by inspection, since $|\psi_1\rangle$ is confined to the $\{|1\rangle, |2\rangle\}$ subspace while $|\psi_2\rangle$ has a component along $|3\rangle$.

Problem 3

(a) The Hamiltonian for our 2-level system has the form

$$H = \hbar \begin{pmatrix} 0 & -\tfrac{1}{2}\left(\chi_{12}\, e^{-i\omega t} + \chi_{21}^{*} e^{i\omega t}\right) \\ -\tfrac{1}{2}\left(\chi_{21}\, e^{-i\omega t} + \chi_{12}^{*} e^{i\omega t}\right) & \omega_{21} \end{pmatrix}$$

Starting from $i\hbar\dot{\rho} = [H,\rho]$ we get $i\hbar\dot{\rho}_{ij} = \sum_k \left(H_{ik}\rho_{kj} - \rho_{ik}H_{kj}\right),\ i,j,k \in \{1,2\}$.
Applying this to the elements of the 2-level density matrix, we get

$$\dot{\rho}_{11} = \frac{1}{i\hbar}\left(H_{11}\rho_{11} - \rho_{11}H_{11} + H_{12}\rho_{21} - \rho_{12}H_{21}\right)$$
$$= \frac{i}{2}\left(\chi_{12}e^{-i\omega t} + \chi_{21}^{*}e^{i\omega t}\right)\rho_{21} - \frac{i}{2}\left(\chi_{21}e^{-i\omega t} + \chi_{12}^{*}e^{i\omega t}\right)\rho_{12}$$

and
$$i\hbar\dot{\rho}_{12} = H_{11}\rho_{12} - \rho_{11}H_{12} + H_{12}\rho_{22} - \rho_{12}H_{22}$$
$$= i\omega_{21}\rho_{12} + \frac{i}{2}\left(\chi_{12}e^{-i\omega t} + \chi_{21}^{*}e^{i\omega t}\right)\left(\rho_{22} - \rho_{11}\right)$$

Now let $\rho_{12} = \tilde{\rho}_{12}e^{i\omega t}$, $\rho_{21} = \tilde{\rho}_{21}e^{-i\omega t}$ and substitute in the above equations. This gives us

$$\dot{\rho}_{11} = \frac{i}{2}\left(\chi_{12}e^{-i\omega t} + \chi_{21}^{*}e^{i\omega t}\right)\rho_{21} - \frac{i}{2}\left(\chi_{21}e^{-i\omega t} + \chi_{12}^{*}e^{i\omega t}\right)\rho_{12}$$
$$\Rightarrow\ \dot{\rho}_{11} = \frac{i}{2}\left(\chi_{12}e^{-2i\omega t} + \chi_{21}^{*}\right)\tilde{\rho}_{21} - \frac{i}{2}\left(\chi_{21} + \chi_{12}^{*}e^{2i\omega t}\right)\tilde{\rho}_{12}$$

$$\dot{\rho}_{12} = \frac{d}{dt}\left(\tilde{\rho}_{12}e^{i\omega t}\right) = \dot{\tilde{\rho}}_{12}e^{i\omega t} + i\omega\tilde{\rho}_{12}e^{i\omega t} = i\omega_{21}\tilde{\rho}_{12}e^{i\omega t} + \frac{i}{2}\left(\chi_{12}e^{-i\omega t} + \chi_{21}^{*}e^{i\omega t}\right)\left(\rho_{22} - \rho_{11}\right)$$
$$\Rightarrow\ \dot{\tilde{\rho}}_{12} = i\left(\omega_{21} - \omega\right)\tilde{\rho}_{12}\,\frac{i}{2}\left(\chi_{12}e^{-i2\omega t} + \chi_{21}^{*}\right)\left(\rho_{22} - \rho_{11}\right)$$

We set $\Delta = \left(\omega_{21} - \omega\right)$, $\chi_{12} = \chi$, $\chi_{21}^{*} = \chi^{*}$, drop the terms $\propto e^{\pm i2\omega t}$, and use $\dot{\rho}_{22} = -\dot{\rho}_{11}$, $\dot{\rho}_{12} = \dot{\rho}_{21}^{*}$. This gives us the desired result

$$\dot{\rho}_{11} = -\frac{i}{2}\left(\chi\rho_{12} - \chi^{*}\rho_{21}\right), \qquad \dot{\rho}_{22} = \frac{i}{2}\left(\chi\rho_{12} - \chi^{*}\rho_{21}\right)$$
$$\dot{\rho}_{12} = i\Delta\rho_{12} + \frac{i\chi^{*}}{2}\left(\rho_{22} - \rho_{11}\right), \qquad \dot{\rho}_{21} = -i\Delta\rho_{21} - \frac{i\chi^{2}}{2}\left(\rho_{22} - \rho_{11}\right)$$

(b) The Density Matrix has 2 real-valued populations and 2 complex-valued coherences, which suggests a total of 6 real-valued variables that must be known in order to specify ρ. However, the constraints $\rho_{22} = 1 - \rho_{11}$ and $\rho_{12} = \rho_{21}^*$ allow us to express 3 of the 6 variables in terms of the other 3. This leaves us with a total of 3 real-valued variables necessary to specify an arbitrary Density Matrix, whether it is pure or mixed.

(c) Major approximations implicit in the above result:

(i) The Electric Dipole Approximation (inherent in the form of *H*)
(ii) The 2-Level Approximation
(iii) The Rotating Wave Approximation

Problem 4

In steady state the Density Matrix Equations reduce to

(i) $\dot{\rho}_{11} = A_{21}\rho_{22} - \frac{i}{2}\left(\chi\rho_{12} - \chi^*\rho_{21}\right) = 0$

(ii) $\dot{\rho}_{22} = -A_{21}\,\rho_{22} + \frac{i}{2}\left(\chi\rho_{12} - \chi^*\rho_{21}\right) = 0$

(iii) $\dot{\rho}_{12} = -\left(\beta - i\Delta\right)\rho_{12} + i\frac{\chi^*}{2}\left(\rho_{22} - \rho_{11}\right) = \dot{\rho}_{21}^* = 0$

We start by solving for the coherences in Equation (iii)

$$\rho_{12} = \frac{i\frac{\chi^*}{2}\left(\rho_{22} - \rho_{11}\right)}{\beta - i\Delta} = i\frac{\chi^*}{2}\frac{\beta + i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right), \qquad \rho_{21} = -i\frac{\chi}{2}\frac{\beta - i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right)$$

From this we get $\chi\rho_{12} - \chi^*\rho_{21} = \frac{i|\chi|^2\beta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right)$

Substituting in Equation (ii), using $\rho_{22} - \rho_{11} = 2\rho_{22} - 1$, and solving for ρ_{22}, we get

$$\rho_{22} = \frac{i}{2A_{21}}\frac{i|\chi|^2\beta}{\Delta^2 + \beta^2}\left(2\rho_{22} - 1\right) \Rightarrow \left(1 + \frac{|\chi|^2\beta / A_{21}}{\Delta^2 + \beta^2}\right)\rho_{22} = \frac{1}{2}\frac{|\chi|^2\beta / A_{21}}{\Delta^2 + \beta^2}$$

$$\Rightarrow \quad \rho_{22} = \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}}$$

Next,

$$\rho_{11} = 1 - \rho_{22} = 1 - \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} = \frac{\Delta^2 + \beta^2 + |\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}}$$

Sanity check: $\rho_{11} + \rho_{22} = 1$. With that we have the steady state solutions

$$\rho_{11}(\infty) = \frac{\Delta^2 + \beta^2 + |\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} \qquad \rho_{22}(\infty) = \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}}$$

$$\rho_{12}(\infty) = i\frac{\chi^*}{2}\frac{\beta + i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right) \qquad \rho_{21}(\infty) = -i\frac{\chi}{2}\frac{\beta - i\Delta}{\Delta^2 + \beta^2}\left(\rho_{22} - \rho_{11}\right)$$