OPTI 544, Problem Set 2
Posted January 27, Due January 6

Electronic Submission only, by email to Ian Marsh (marshian@email.arizona.edu)

I

In the rotating wave approximation the equation of motion for the state of a two-level atom can be reduced to the form

$$i\dot{c}_1 = -\tfrac{1}{2}\chi^* c_2 ,$$

$$i\dot{c}_2 = \Delta c_2 - \tfrac{1}{2}\chi c_1 ,$$

where $\chi = \langle 2|-\hat{\vec{p}}\cdot\vec{E}/\hbar|1\rangle$ is the (complex) Rabi frequency, and $\Delta = \omega_{21} - \omega$ is the detuning from resonance.

(a) Find the general solution to these equations. Then choose its free parameters to fit the initial conditions $c_1(t=0)=0$, $c_2(t=0)=1$.

Note: If you don't remember how to find the general solution to this kind of differential equation and therefore have a hard time answering part (a), you may instead verify that the solution from the class notes does in fact satisfy the equations above. Then try to guess how the solution will change for the initial conditions given here.

(b) Plot the probabilities for finding the atom in each of the two states as a function of time, for $\Delta = 0$, $\Delta = \chi$ and $\Delta = 5\chi$. Take care to show everything to scale. (If possible, make the plots on a computer.)

(c) The equations of motion for the probability amplitudes c_i correspond to a time dependent Schrödinger equation governed by a time independent Hamiltonian H. Find the eigenstates (dressed states) of H as function of χ and Δ. Use the method from Chapter IV, complement B-IV in Cohen-Tannoudji, and restrict yourself to the case where χ is real and $\Delta < 0$. Derive simpler, approximate expressions for the dressed states in the limits $|\chi| << |\Delta|$ and $\chi| >> |\Delta|$.

Note: If you don't have a copy of Cohen-Tannoudji, let me know right away and I will get the relevant pages to you.

II

(a) From the eq. of motion for the state vector $\underline{a} = (a_1, a_2)$, assuming $\vec{E}$ and $\vec{p}_{12}$ are real, derive the equation of motion for the expectation value $\langle\hat{\vec{p}}\rangle$ of the dipole operator for a 2-level atom. Under what conditions does it resemble the equation of motion for the dipole $\vec{p}$ of the Lorentz atom?

(b) Write down a general expression for the time-dependent expectation value $\langle\hat{\vec{p}}\rangle$ for the atomic dipole, getting $\underline{a}$ from the solution for the time dependent state vector $\underline{c}$ found in I(a) above. Hint: Do NOT attempt to solve the differential equation for $\langle\hat{\vec{p}}\rangle$ from II(a)!

(c) Determine the frequencies radiated by $\langle\hat{\vec{p}}\rangle$. Compare to the frequencies radiated by the classical dipole $\vec{p}$ in the Lorentz atom.

(d) Discuss how the spectrum radiated by $\langle \hat{\vec{p}} \rangle$ can be understood in terms of a dipole oscillating at the frequency of the driving field, but with an amplitude which is modulated by Rabi oscillations between the ground and excited states.

Hint: If you are not familiar with the concept of amplitude modulation, take a look here: http://www.youtube.com/watch?v=bb0bzCe4AOc

III

Bonus Problem - Not included in HW Score. I will post the solution in a bit, and let you decide if you want to attempt it on your own. If not, read through the official solution and pay attention to the similarity between the conservation of angular and linear momentum and the selection rules that follow.

In the following we consider a two-level atom interacting with a monochromatic plane wave $\vec{E}(t,z) = \vec{\varepsilon} E_0 e^{-i(\omega t - kz)}$ of light. Allowing the atom to move along the z-axis only, the Hamiltonian for the atom is the sum of internal and kinetic energy, $\hat{H}_A = \hat{H}_0 + \hat{P}^2/2M$, where $\hat{P}$ is the (center-of-mass) momentum along z. The eigenstates of $\hat{H}_A$ are joint eigenstates of internal energy and momentum,

$$|1\rangle|P_1 = \hbar k_1\rangle = |1,k_1\rangle, \qquad |2\rangle|P_2 = \hbar k_2\rangle = |2,k_2\rangle.$$

The electric-dipole interaction between atom and light is $\hat{V} = -\hat{\vec{p}} \cdot \vec{\varepsilon} E(t) e^{ik\hat{Z}}$, where the operator $\hat{\vec{p}} \cdot \vec{\varepsilon} E(t)$ acts solely on the internal degrees of freedom, $\langle j | \hat{\vec{p}} \cdot \vec{\varepsilon} E(t) | l \rangle = \hbar \chi_{jl}(t)$ where $j,l = 1,2$. Similarly, the operator $e^{ik\hat{Z}}$ acts solely on the center-of-mass degree of freedom. We are looking for selection rules for transitions between (linear) momentum states, in a manner inspired by the way we found selection rules for dipole transitions between angular momentum states.

(a) Write down an expression for the matrix elements $V_{jk_j,lk_l} = \langle j,k_j | -\hat{\vec{p}} \cdot \vec{\varepsilon} E(t) e^{ik\hat{Z}} | l,k_l \rangle$ in terms of the matrix elements $\hbar \chi_{jl}(t)$ and an overlap integral that involves the coordinate representation of the operator $e^{ik\hat{Z}}$ and the momentum wavefunctions $\psi_{P=\hbar k_q}(z)$.

(b) What does the result in (a) tell us about the change in momentum when the atom goes from the ground state to the excited state? What does this tell us about the momentum of light?

(c) Assume the atom starts in a state $|1,k_1\rangle$, that the plane wave is turned on at $t=0$, and that it is resonant with an allowed transition to the state $|2,k_2\rangle$. Find an expression for the expectation value $\langle \hat{P}(t) \rangle$ of the momentum. Make a plot of this quantity as function of time, being careful to indicate on the axes both the timescale for the evolution and the range of momentum that is observed.

Hint: Plane waves are orthogonal functions: $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} (e^{ikz})^* \, e^{ik_0 z} \, dz = \delta(k - k_0)$