OPTI 544 Final Exam, May 6, 2022 Jessen

Solution Set

Problem I

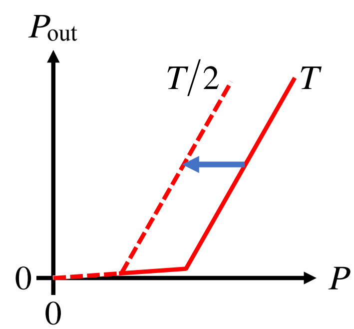
Pure states:	(4), (5)
Mixed states:	(3) and (6)
Unphysical:	(1) and (2) (first is non-Hermitian, second is not unit trace)

Problem II

(a) The plot shows the threshold behavior characteristic of lasers. Below a threshold pumping rate, the roundtrip loss exceeds the roundtrip gain in the cavity, and the intracavity photon flux decays exponentially until it reaches a level that corresponds to spontaneous emission into the lasing mode. That is the section where the output power grows slowly with the pump rate. Above threshold, the intracavity photon flux grows until saturation sets in. At that point, the steady-state gain equals the threshold gain, and the inversion equals the threshold inversion. Increasing P puts more atoms in the upper lasing level, and stimulated emission rapidly converts these excitations into photons in the lasing mode. At that point the output power grows steeply with the pumping rate.

(b) The threshold gain and threshold inversion, $g_t = \sigma \Delta N_t$, is a property of the cavity and the laser wavelength, and those quantities do not depend on the laser pumping scheme. The pumping rate required to reach threshold *does* depend on the pumping scheme, and is typically lower for a 4-level scheme than for a 3-level scheme. But to distinguish one from the other, we would need additional quantitative information that is not available in the plot.

(c) If we cut losses due to transmission through the output coupler by a factor of 2, the threshold pumping rate drops by a factor of 2. The behaviors above and below threshold are the same as in (a). In particular, far above threshold where stimulated emission outstrips spontaneous decay by a large factor, the excitation of an atom in the gain medium is immediately converted into a photon which exits the cavity through the output coupler. That means the *slope efficiency* is the same, as indicated by the dashed line in the figure.

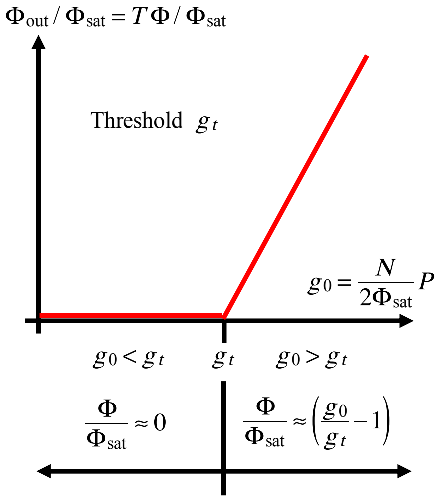


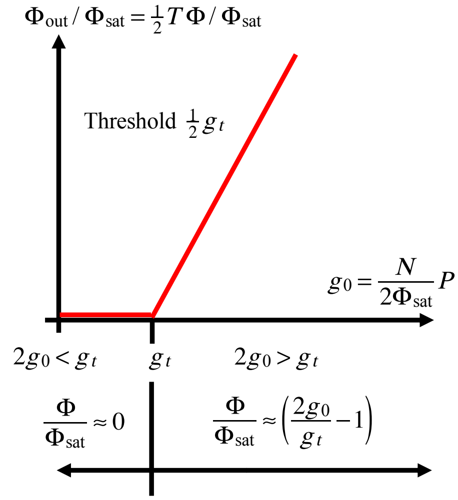
Note: Formally, when $g_0 >> g_t$ (operating the laser well above threshold) we have

$$g = g_t = \frac{g_0}{1 + \Phi / \Phi_{\text{sat}}} \quad \Rightarrow \quad \frac{\Phi}{\Phi_{\text{sat}}} \approx \left(\frac{g_0}{g_t} - 1 \right)$$

where g_0 is the small signal gain, g_t is the threshold gain, g is the saturated gain, Φ is the intracavity photon flux, and Φ_{sat} is the saturation flux. In trying to visualize how this plays out, it is also helpful to note that for both 3- and 4-level pumping schemes well above threshold and with the usual hierarchies of rates, we have $g_0 \propto P$.

In the visualization below, the figure on the left shows the situation with threshold gain g_t. The figure on the right illustrates what happens when we substitute $g_t \to \frac{1}{2} g_t$, which is functionally equivalent to substituting $g_0 \to 2g_0$. The result is that threshold is reached at half the pumping rate, the intracavity flux Φ is doubled, and the output flux Φ_{out} is unchanged.





Needless to say, only a conceptual discussion along the lines in part (c) is expected in the context of the exam.

Problem III

(a) As derived in class, the output state is $|\Psi_{\text{out}}\rangle = (t\,\hat{a}_3^{+} + r\hat{a}_4^{+})|0\rangle = t|1\rangle_3|0\rangle_4 + r|0\rangle_3|1\rangle_4$.

(b) As shown in Part 1(a) of Problem Set 8, a coherent state input, $|\Psi_{\text{in}}\rangle = |\alpha\rangle_1|0\rangle_2$, gives us an output $|\Psi_{\text{out}}\rangle = |t\alpha\rangle_3|r\alpha\rangle_4$. This is a product state, with no photon number-mode entanglement, so the output in port 3 is the coherent state $|\Psi_{\text{out}}\rangle = |t\alpha\rangle_3$. Generalizing to any lossy medium with transmission coefficient *t*, we see that the input-output map is $|\alpha\rangle \longrightarrow |t\,\alpha\rangle$. Thus the coherent state remains a coherent state, though with reduced amplitude.

Problem IV

(a) We can deposit a photon in an empty cavity as follows. Put the atom in the excited state, then shoot it though the cavity with a velocity such that the atom-cavity interaction integrated along the path corresponds to a π pulse. With *m* photons initially in the cavity the Rabi frequency along the path increases by a factor $\sqrt{m+1}$, so successive atoms will need to cross the cavity at progressively higher velocity for the atom-cavity interaction to constitute a π pulse.

(b) The position and time dependence of the Rabi frequency along the atom's path is

$$g(z) = 2g\sqrt{n+1}\; e^{-z^2/2\sigma^2} = 2g\sqrt{n+1}\; e^{-(vt)^2/2\sigma^2} \;=\; 2g\sqrt{n+1}\; e^{-t^2/2(\sigma/v)^2}$$

To deposit exactly one photon in the initially empty cavity ($n = 0$), we need

$$2g\int_{-\infty}^{\infty} e^{-t^2/2(\sigma/v)^2} dt = 2g\sqrt{2\pi}\,\frac{\sigma}{v} \;=\; \pi \quad\Rightarrow\quad v = 2g\sigma\sqrt{\frac{2}{\pi}}$$