OPTI 544 Solution Set 9, Spring 2022

Problem 1

(a) The vacuum Rabi frequency is $g = -\dfrac{\vec{p}_{21} \cdot \vec{\varepsilon}_k \mathscr{E}}{\hbar}$, where $\mathscr{E}_k = \sqrt{\dfrac{\hbar\omega_k}{\varepsilon_0 V}}$.

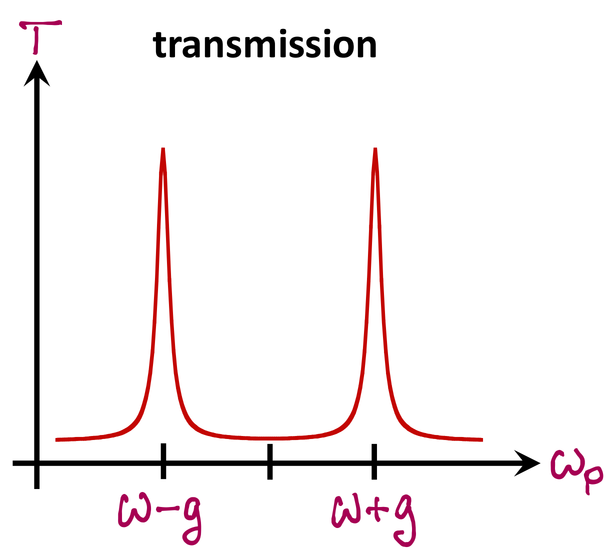
The field is parallel to the dipole so that $\vec{p}_{21} \cdot \vec{\varepsilon}_k = |\vec{p}_{21}| = p_{21}$, where $A_{21} = \dfrac{8\pi^2 p_{21}^2}{3\varepsilon_0 \hbar \lambda^3}$.

Putting this together gives us the vacuum Rabi frequency:

$$g = \frac{1}{\hbar}\sqrt{\frac{3\varepsilon_0 \hbar \lambda^3 A_{21}}{8\pi^2}}\sqrt{\frac{\hbar\omega_k}{\varepsilon_0 V}} = \sqrt{\frac{3c\lambda^2 A_{21}}{4\pi V}}$$
$$= \sqrt{\frac{3 \times 3 \times 10^8 m/s \times (852 \times 10^{-9} m)^2 \times 3.28 \times 10^7 /s}{4\pi \times 100 \times 10^{-6} m \times \pi (2 \times 10^{-6} m)^2}} = 1.165 \times 10^9 /s$$

We see that g exceeds the rate of energy loss due to decay of the atomic excitation by more than an order of magnitude. This means the system is in a regime where we can see coherent vacuum Rabi oscillations.

(b) The probe transmission is going to show vacuum Rabi splitting. Borrowing a sketch from the class notes, this looks as follows;



Problem II

(a) The modified Jaynes-Cummings Hamiltonian is

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\hbar\omega\left(\hat{\sigma}_z^{(1)} + \hat{\sigma}_z^{(2)}\right)}_{\hat{H}_0} + \underbrace{\hbar g\left(\hat{\sigma}_+^{(1)}\hat{a} + \hat{\sigma}_-^{(1)}\hat{a}^\dagger + \hat{\sigma}_+^{(2)}\hat{a} + \hat{\sigma}_-^{(2)}\hat{a}^\dagger\right)}_{\hat{H}_{AF}}$$

(b) Eigenstates of $\hat{H}_0$:

state	energy
$\vert 1,1,0\rangle$	$-\hbar\omega$
$\vert 1,1,1\rangle$, $\vert 2,1,0\rangle$, $\vert 1,2,0\rangle$	0

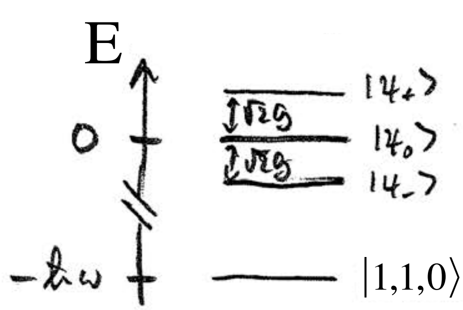
In basis $\{\vert 1,1,0\rangle, \vert 1,1,1\rangle, \vert 2,1,0\rangle, \vert 1,2,0\rangle\}$ we have $\hat{H}_{AF} = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & g & g \\ 0 & g & 0 & 0 \\ 0 & g & 0 & 0 \end{pmatrix}$

The upper left 1×1 represents the ground state which is not coupled and has eigenvalue $\lambda = 0$.

The eigenvalues of the 3×3 block are found by setting

$$\begin{vmatrix} -\lambda & g & g \\ g & -\lambda & 0 \\ g & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda g^2 = \lambda\left(\lambda^2 - 2g^2\right) = 0 \;\Rightarrow\; \lambda = 0, \pm\sqrt{2}g$$

Level diagram:

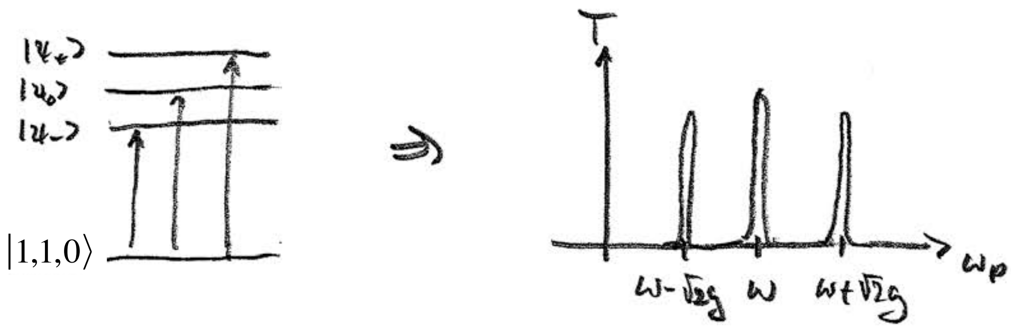


The eigenvalues of $\hat{H}$ are thus

$\lambda = -\hbar\omega$

$\lambda = 0, \pm\sqrt{2}\hbar g$ $\lambda = 0, \pm\sqrt{2}\hbar g$

(c)



We expect to see vacuum Rabi splitting. Transmission will occur when the probe is resonant with transitions

(d) Restricting to the single-excitation manifold and setting $|\psi_0\rangle = a|1,1,1\rangle + b|2,1,0\rangle + c|1,2,0\rangle$, the relevant eigenvalue problem is

$$\begin{pmatrix} 0 & g & g \\ g & 0 & 0 \\ g & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} gb + gc = 0 \\ ga = 0 \\ |a|^2 + |b|^2 + |c|^2 = 1 \end{cases}$$

From this we seee that $|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|2,1,0\rangle - |2,1,0\rangle\right)$ is a solution.

The probe has to add a photon to the cavity and theerfore can only couple the ground state $|1,1,0\rangle$ to excited states that have an admixture of $|1,1,1\rangle$, i. e., the transition $|1,1,0\rangle \to |\psi_0\rangle$ is forbidden and we do not see transmission at $\omega_p = \omega$. The transmission spectrum thus looks like the single atom case, but with a vacuum Rabi splitting that is a factor of $\sqrt{2}$ larger.

<u>Note:</u> It is straighforward to find the other two states in the single-excitation manifold,

$$|\psi_\pm\rangle = \tfrac{1}{2}\left(\pm|1,1,1\rangle + \tfrac{1}{\sqrt{2}}|2,1,0\rangle + \tfrac{1}{\sqrt{2}}|1,2,0\rangle\right)$$

These states have a $|1,1,1\rangle$ component and are therefore coupled to the $|1,1,0\rangle$ state by the probe. This confirms that we will still see transmission at $\omega_p = \omega \pm \sqrt{2}g$.