Quantum Electrodynamics – QED

Starting point: Maxwells Equations

(1) $\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$

(2) $\nabla \cdot \vec{B}(\vec{r},t) = 0$

(3) $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$

(4) $\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

Implicit: Charges & Fields in Vacuum
No "medium response"

Same issue as with our introductory example:

Maxwells eqs are non-local

We need to put the classical description in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

(1) $i\vec{k} \cdot \vec{\mathcal{E}}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k} \cdot \vec{\mathcal{B}}(\vec{k},t) = 0$

(3) $i\vec{k} \times \vec{\mathcal{E}}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t)$

(4) $i\vec{k} \times \vec{\mathcal{B}}(\vec{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t) + \frac{1}{\varepsilon_0 c^2}\vec{\mathcal{J}}(\vec{k},t)$

Fourier Transform:
$$\begin{cases} \nabla \cdot \vec{G} \rightleftarrows i\vec{k} \cdot \vec{g} \\ \nabla \times \vec{G} \rightleftarrows i\vec{k} \times \vec{g} \end{cases}$$

Note: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

Quantum Electrodynamics – QED

Free Fields - Switch to Fourier Domain

(1) $i\vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k}\cdot\vec{\mathcal{B}}(\vec{k},t) = 0$

(3) $i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t)$

(4) $i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{k},t)$

Fourier Transform:

$$\begin{cases} \nabla\cdot\vec{G} \rightleftarrows i\vec{k}\cdot\vec{G} \\ \nabla\times\vec{G} \rightleftarrows i\vec{k}\times\vec{G} \end{cases}$$

Note: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)} + \vec{\mathcal{B}}_{\perp}(\vec{k},t) \qquad \text{MEq (2)}$$

Entirely Transverse

Note:

- $-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k}$ $\times$ the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)} + \vec{\mathcal{B}}_{\perp}(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

Note:
- $-\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}$ **is the projection of** $\vec{\mathcal{E}}$ **onto** $\vec{k}$
- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k} \times$ **the projection of** $\vec{\mathcal{E}}$ **onto** $\vec{k}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $$\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

(6a) $$\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$$

inverse FT

(5b) $$\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_{\perp}(\vec{r},t)$$

(6b) $$\frac{\partial}{\partial t}\vec{E}_{\perp}(\vec{r},t) = c^2\nabla\times\vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{r},t)$$

Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t) = \cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)} + \vec{\mathcal{B}}_{\perp}(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

Note: $\left\{ \begin{array}{l} -\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}} \text{ is the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \\ \vec{\mathcal{E}}_{\parallel} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \end{array} \right.$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = \frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

$\boxed{\text{Only } \vec{\mathcal{E}}_{\perp} \text{ and } \vec{\mathcal{B}}_{\perp} \text{ are new degrees of freedom beyond the particles -> } \underline{\text{Free Fields}}}$

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_{\perp}(\vec{k},t)$

(6a) $\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$

inverse FT

(5b) $\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_{\perp}(\vec{r},t)$

(6b) $\frac{\partial}{\partial t}\vec{E}_{\perp}(\vec{r},t) = c^2\nabla\times\vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{r},t)$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_{\perp}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}_{\perp}(\vec{r},t)$$

Quantum Electrodynamics – QED

Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_\perp(\vec{k},t)$

(6a) $\frac{\partial}{\partial t}\vec{\mathcal{E}}_\perp(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_\perp(\vec{k},t)$

inverse FT

(5b) $\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_\perp(\vec{r},t)$

(6b) $\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = c^2\nabla\times\vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_\perp(\vec{r},t)$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t^2}\vec{j}_\perp(\vec{r},t)$$

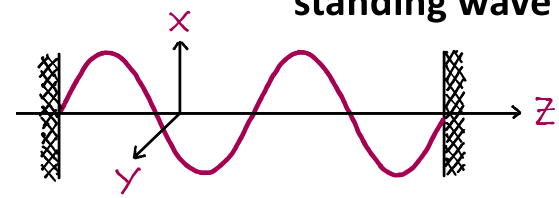
Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$

standing wave



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\varepsilon}_x E_x(z,t)$ and expand

fiducial mass

(7) $E_x(z,t) = \sum_j A_j q_j(t)\sin(k_j z)\,,\quad A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}$

MEq (4) w/no charges

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\varepsilon}_x\frac{1}{c^2}\sum_j A_j\dot{q}_j(t)\sin(k_j t)$$

$$= \vec{\varepsilon}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right) = -\vec{\varepsilon}_x\frac{\partial B_y}{\partial z}$$

$\mathcal{B}$ transverse ⇨ $B_z = 0$

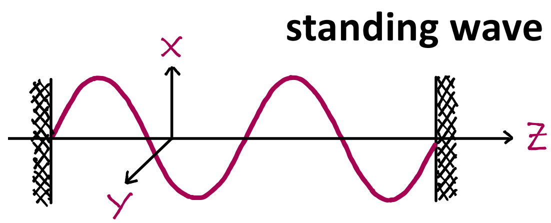
Quantum Electrodynamics – QED

Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\varepsilon}_x E_x(z,t)$ and expand

fiducial mass

$$\boxed{(7)\quad E_x(z,t) = \sum_j A_j q_j(t)\sin(k_j z)\,,\quad A_j = \sqrt{\frac{2\omega_j m_j}{\varepsilon_0 V}}}$$

MEq (4) w/no charges

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\varepsilon}_x\frac{1}{c^2}\sum_j A_j\dot{q}_j(t)\sin(k_j t)$$

$$= \vec{\varepsilon}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right) = -\vec{\varepsilon}_x\frac{\partial B_y}{\partial z}$$

$\mathcal{B}$ transverse $\Rightarrow$ $B_z = 0$

From Eq. (5a) we see that

$$\vec{B}\perp\vec{E},\vec{\varepsilon}_z \Rightarrow \vec{B}(z,t) = \vec{\varepsilon}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j\frac{A_j}{c^2}q_j(t)\sin(k_j z)$$

$$\boxed{(8)\quad B_y(z,t) = \sum_j\frac{A_j}{k_j c^2}\dot{q}_j(t)\cos(k_j z)}$$

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\varepsilon_0 A}{2}\int_0^L dz\left(|\vec{E}|^2 + c^2|\vec{B}|^2\right) =$$

$$\frac{\varepsilon_0 A}{2}\int_0^L dz\sum_j\left[A_j^2 q_j(t)^2\sin^2(k_j z) + \frac{A_j^2}{k_j}\dot{q}_j(t)^2\cos^2(k_j z)\right]$$

Quantum Electrodynamics – QED

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\mathcal{E}}_z \;\Rightarrow\; \vec{B}(z,t) = \vec{\mathcal{E}}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j \frac{A_j}{c^2} q_j(t) \sin(k_j z)$$

$$\Downarrow$$

$$\boxed{(8) \qquad B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)}$$

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\varepsilon_0 A}{2} \int_0^L dz \left(|\vec{E}|^2 + c^2 |\vec{B}|^2 \right) =$$

$$\frac{\varepsilon_0 A}{2} \int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j} \dot{q}_j(t)^2 \cos^2(k_j z) \right]$$

Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{2\omega_j^2 m_j}{\varepsilon_0 V}$ we finally get

$$\boxed{\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]}$$

Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\varepsilon_0 A}{2} \int_0^L dz \left(c^2 |\vec{B}|^2 - |\vec{E}|^2 \right)$$

$$= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

Check $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \;\Rightarrow\; \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_\perp(\vec{r}, t) = 0 \;\Rightarrow\; \ddot{q}_j + \omega_j^2 q_j = 0$$

Quantum Electrodynamics – QED

Integrating over the Cavity volume

$$A\int_0^L dz\,\sin^2(k_j z) = A\int_0^L dz\,\cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \dfrac{2\omega_j^2 m_j}{\varepsilon_0 V}$ **we finally get**

$$\boxed{\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2\right]}$$

Lagrangian for the Classical Field

$$\boxed{\begin{aligned}\mathcal{L} &= \frac{\varepsilon_0 A}{2}\int_0^L dz\left(c^2|\vec{B}|^2 - |\vec{E}|^2\right)\\ &= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2\right]\end{aligned}}$$

Check $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_j} - \dfrac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

Quantum Electrodynamics – QED

Integrating over the Cavity volume

$$A\int_0^L dz\,\sin^2(k_j z) = A\int_0^L dz\,\cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{2\omega_j^2 m_j}{\varepsilon_0 V}$ we finally get

$$\boxed{\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2\right]}$$

Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\varepsilon_0 A}{2}\int_0^L dz\left(c^2|\vec{B}|^2 - |\vec{E}|^2\right)$$

$$= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2\right]$$

☑

Check $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_j} - \frac{\partial\mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

And Finally:

Conjugate Momentum $\boxed{p_j = \frac{\partial\mathcal{L}}{\partial\dot{q}_j} = m_j\dot{q}_j}$

As before, a collection of Harmonic Oscillators, ready for quantization!

Quantum Electrodynamics – QED

To summarize so far…

$$E_x(z,t) = \sum_j A_j\, q_j(t) \sin(k_j z)$$

$$A_j = \sqrt{\frac{\omega_j^2 m_j}{2\varepsilon_0 V}}$$

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\, \dot{q}_j(t) \cos(k_j z)$$

$$\mathcal{L} = \frac{\varepsilon_0 A}{2}\int_0^L dz\left(c^2|\vec{B}|^2 - |\vec{E}|^2\right) = \sum_j\left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j q_j^2\right]$$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j$$

$$\mathcal{H} = \sum_j\left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2\right]$$

Classical Fields

Dimensionless Field Variables:

$$Q_j = q_j / q_{o,j}\,, \quad q_{o,j} = \sqrt{2\hbar / m_j \omega_j}$$

$$P_j = p_j / p_{o,j}\,, \quad p_{o,j} = \sqrt{2\hbar m_j \omega_j}$$

$$\alpha_j(t) = Q_j(t) + i\,P_j(t) = \alpha_j(0)\, e^{-i\omega_j t}$$

$$E_x(z,t) = \sum_j A_j\, q_j(t)\sin(k_j z)\,, \qquad \mathcal{E}_j = A_j\, q_{o,j} = \sqrt{\frac{\hbar\omega_j}{\varepsilon_0 V}}$$

$$= \sum_j \mathcal{E}_j\left[\alpha_j(t) + \alpha_j^*(t)\right]\sin(k_j z)$$

field
"per photon"

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\, \dot{q}_j(t)\cos(k_j z)$$

$$= -\frac{i}{c}\sum_j \mathcal{E}_j\left[\alpha_j(t) - \alpha_j^*(t)\right]\cos(k_j t)$$

Quantum Electrodynamics – QED

Classical Fields

Dimensionless Field Variables:

$$Q_j = q_j / q_{o,j} \, , \quad q_{o,j} = \sqrt{2\hbar / m_j \omega_j}$$

$$P_j = p_j / p_{o,j} \, , \quad p_{o,j} = \sqrt{2\hbar m_j \omega_j}$$

$$\boxed{\alpha_j(t) = Q_j(t) + i\, P_j(t) = \alpha_j(0)\, e^{-i\omega_j t}}$$

$$E_x(z,t) = \sum_j A_j\, q_j(t) \sin(k_j z) \, , \quad \mathcal{E}_j = A_j\, q_{o,j} = \sqrt{\frac{\hbar\omega_j}{\varepsilon_o V}}$$

$$= \sum_j \mathcal{E}_j \left[\alpha_j(t) + \alpha_j^*(t)\right] \sin(k_j z)$$

field
"per photon"

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\, \dot{q}_j(t) \cos(k_j z)$$

$$= -\frac{i}{c} \sum_j \mathcal{E}_j \left[\alpha_j(t) - \alpha_j^*(t)\right] \cos(k_j t)$$

Standard Quantization Procedure

$$q_j \rightarrow \hat{q}_j$$

$$p_j \rightarrow \hat{p}_j \, , \qquad [\hat{q}_j, \hat{p}_{j'}] = i\hbar\, \delta_{jj'}$$

To be continued…