OPTI 544 Solution Set 6, Spring 2022

Problem 1

(a) **For a 3-level system**, with the rates indicated in the notes and lecture slides and setting $N_3 = 0$, we end up with the following Rate Equations :

(i) $\dot{N}_1 = -PN_1 + \Gamma_{21}N_2 + \sigma\Phi\,(N_2 - N_1)$

(ii) $\dot{N}_2 = PN_1 - \Gamma_{21}N_2 - \sigma\Phi\,(N_2 - N_1)$

Setting Eq. (i) equal to zero, multiplying out and solving for N_1, we get

$$-(P+\sigma\Phi)N_1 + (\Gamma_{21}+\sigma\Phi)N_2 = 0 \;\Rightarrow\; N_1 = \frac{\Gamma_{21}+\sigma\Phi}{P+\sigma\Phi}N_2$$

Then $N \;=\; N_2 + N_1 \;=\; \left(1+\frac{\Gamma_{21}+\sigma\Phi}{P+\sigma\Phi}\right)N_2 \;=\; \left(\frac{P+\Gamma_{21}+2\sigma\Phi}{P+\sigma\Phi}\right)N_2$

and

$$\boxed{\Delta N_{3level} = 2N_2 - N \;=\; \left(\frac{2(P+\sigma\Phi)}{P+\Gamma_{21}+2\sigma\Phi}-1\right)N \;=\; \frac{(P-\Gamma_{21})}{P+\Gamma_{21}+2\sigma\Phi}N}$$

For a 4-level system, with the rates indicated in the notes and lecture slides and setting $N_3 = 0$, we end up with the following Rate Equations :

(i) $\dot{N}_0 = -PN_0 + \Gamma_{10}N_1$

(ii) $\dot{N}_1 = -\Gamma_{10}N_1 + \Gamma_{21}N_2 + \sigma\Phi(N_2 - N_1)$

(iii) $\dot{N}_2 = PN_0 - \Gamma_{21}N_2 - \sigma\Phi(N_2 - N_1)$

Set Eq. (i) $= 0 \;\Rightarrow\;$ $-PN_0 + \Gamma_{10}N_1 = 0 \;\Rightarrow\; N_0 = \frac{\Gamma_{10}}{P}N_1$.

Set Eq. (ii) $= 0 \;\Rightarrow\;$ $-(\Gamma_{10}+\sigma\Phi)N_1 + (\Gamma_{21}+\sigma\Phi)N_2 = 0 \;\Rightarrow\; N_1 = \frac{\Gamma_{21}+\sigma\Phi}{\Gamma_{10}+\sigma\Phi}N_2$

Combining this, we get $N \;=\; N_2 + N_1 + N_0 \;=\; N_2 + \left(1+\frac{\Gamma_{10}}{P}\right)N_1$

$$= \left(1+\frac{(1+\Gamma_{10}/P)(\Gamma_{21}+\sigma\Phi)}{\Gamma_{10}+\sigma\Phi}\right)N_2 = \frac{\Gamma_{10}+\sigma\Phi+(1+\Gamma_{10}/P)(\Gamma_{21}+\sigma\Phi)}{\Gamma_{10}+\sigma\Phi}N_2$$

And finally

$$\Delta N = N_2 - N_1 = \left(1 - \frac{\Gamma_{21} + \sigma\Phi}{\Gamma_{10} + \sigma\Phi}\right) N_2 = \frac{\Gamma_{10} - \Gamma_{21}}{\Gamma_{10} + \sigma\Phi} N_2 = \frac{\Gamma_{10} - \Gamma_{21}}{\Gamma_{10} + \sigma\Phi + (1 + \Gamma_{10}/P)(\Gamma_{21} + \sigma\Phi)} N$$

Multiply top and bottom with P and rearrange to get

$$\boxed{\Delta N_{4level} = \frac{P(\Gamma_{10} - \Gamma_{21})}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21} + (2P + \Gamma_{10})\,\sigma\Phi} N}$$

(b) **For a 3-level system** we have a small-signal gain (set $\Phi = 0$ in expression for ΔN)

$$\boxed{g_0(\omega) = \sigma\Delta N = \frac{\sigma(P - \Gamma_{21})N}{P + \Gamma_{21}}}$$

Now let $\Phi > 0$ and $g(\omega) = \sigma\Delta N$. Plug in ΔN_{3level} from part (a) and divide top and bottom with $P + \Gamma_{21}$ to get

$$\boxed{g(\omega) = \frac{g_0(\omega)}{1 + \Phi/\Phi_{sat}}, \qquad \Phi_{sat} = \frac{P + \Gamma_{21}}{2\sigma}}$$

For a 4-level system we follow the steps above, using ΔN_{4level} and dividing top and bottom with $P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}$, to get

$$\boxed{g(\omega) = \frac{g_0(\omega)}{1 + \Phi/\Phi_{sat}}, \quad g_0(\omega) = \frac{\sigma P(\Gamma_{10} - \Gamma_{21})\,N}{P(\Gamma_{01} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}, \quad \Phi_{sat} = \frac{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}{(2P + \Gamma_{10})\sigma}}$$

Problem 2

(a) The damping rate κ is the fractional loss in intensity per unit time, here given by the fractional loss per roundtrip in the cavity divided by the roundtrip time,

$$\kappa = (1 - R_1 R_2)\,\frac{2}{2L} = (1 - 1\times 0.99)\,\times\frac{3\times 10^8\,\text{m/s}}{2\times 0.1\,\text{m}} \;=\; \underline{1.50\times 10^7/\text{s}}$$

Then $$g_t = \frac{\kappa}{c} = \frac{1-R_1R_2}{2L} = \frac{0.01}{2\times 0.1\,\text{m}} = \underline{0.05/\text{m}}$$

$$\Delta N_t = \frac{g_t}{\sigma} = \frac{1-R_1R_2}{2L}\frac{2\pi}{\lambda^2} = \frac{0.01\times\pi}{0.1\,\text{m}\times\left(1\times 10^{-6}\,\text{m}\right)^2} = \underline{3.14\times 10^{11}/\text{m}^3}$$

(b) For lasing above threshold the saturated gain is equal to the threshold gain, and the inversion is clamped at ΔN_t. Using the result from Problem 1 and observing the hierarchy of rates, we get

$$\Delta N_t \;=\; \frac{P(\Gamma_{10} - \cancel{\Gamma_{21}})N}{P(\Gamma_{10} + \cancel{\Gamma_{21}}) + \cancel{\Gamma_{10}\Gamma_{21}} + (\cancel{2}P + \Gamma_{10})\,\sigma\Phi} \;\approx\; \frac{PN}{P + \sigma\Phi}$$

$$\Phi \;=\; P\frac{(N - \Delta N_t)}{\sigma\Delta N_t} \;\approx\; P\frac{N}{g_t}$$

The output power is $P_{out} = T_2\hbar\omega\Phi A = \dfrac{T_2 2\pi\hbar c}{\lambda}\dfrac{PN}{g_t}A$

$$= \frac{0.01\times\left(2\pi\times 1.05\times 10^{-34}\text{J}\cdot\text{s}\right)\times 3\times 10^8\text{m}\cdot\text{s}^{-1}\times\left(0.5\times 10^{-6}\text{m}^2\right)\times 10^8\text{s}^{-1}\times 10^{16}\text{m}^{-3}}{10^{-6}\text{m}\times 0.05\text{m}^{-1}}$$

$$= \; 20\times 10^{-3}\,\text{W/s} \;=\; \underline{20\text{mW}}$$