Quantum States of the Quantized Field

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t)$ @ $t=0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k (\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\boxed{\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a}+\hat{a}^+) \qquad \lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)}$$

Homework Problem
(voluntary)

$$\boxed{\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega')T/2} \frac{\sin[(\omega-\omega')T/2]}{(\omega-\omega')/2}}$$

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ drives $\hat{E}(t)$

classical dipole + quantum fluctuations

coherent state + quantum fluctuations

Drive for $t \in [0,T]$

$\alpha(t) = \alpha(T) e^{-i\omega(t-T)}$ for $t > T$

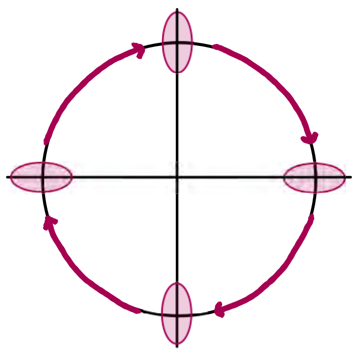
Quantum States of the Quantized Field

Squeezed States

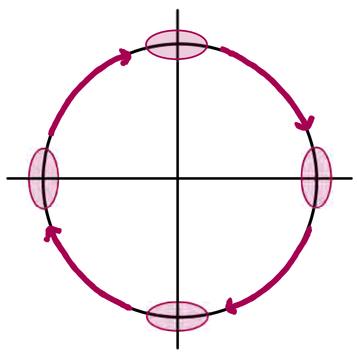
Minimum uncertainty states w/assymmetry

$$\Delta X \Delta Y = 1/4, \quad \Delta X(t) \neq \Delta Y(t)$$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium

Quantum States of the Quantized Field

Odds and Ends – Thermal States

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_B T}\right]$$

$$\hat{\rho} = \sum_n P(n)|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-E_n/k_B T}|n\rangle\langle n|$$

$$= (1-q)\sum_n q^n |n\rangle\langle n|, \quad q = e^{-\hbar\omega/k_B T}$$

Mean Photon Number:

$$\bar{n} = \mathrm{Tr}(\hat{\rho}\hat{N}) = \sum_{k,n}\langle k|(1-q)q^n|n\rangle\langle n|\hat{N}|k\rangle$$

$$= (1-q)\sum_n n q^n = \frac{q}{1-q}$$

Photon Number Uncertainty:

$$\langle \hat{N}^2\rangle = (1-q)\sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

$$\Delta n^2 = \langle \hat{N}^2\rangle - \langle \hat{N}\rangle^2$$

$$= \frac{q^2+q}{(1-q)} - \frac{q^2}{(1-q)^2} = \frac{q}{(1-q)^2}$$

$$\bar{n} = \frac{q}{1-q}$$

Coherent State limit

$$\Delta n = \frac{\sqrt{q}}{1-q} = \sqrt{\bar{n}(\bar{n}+1)} \geq \sqrt{\bar{n}}$$

Optical Frequencies, Room Temperature:

$$\lambda = 1\,\mu m, \qquad T = 300\,K$$

$$q = 6.5\times 10^{-6}, \quad \bar{n} \sim 10^{-6}$$

Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

Use $[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$ to show

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t} T_\alpha$$

$$\Rightarrow \hat{T}_\alpha \hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha \hat{a}\hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$$

From this we get

$$\hat{E}_\perp' = \hat{T}_\alpha \hat{E}_\perp \hat{T}_\alpha^+ = \hat{T}_\alpha(\mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C.)\hat{T}_\alpha^+$$

$$= \mathcal{E}_k \hat{a} e^{i\vec{k}\cdot\vec{r}} + H.C. + \mathcal{E}_k \alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

$$= \hat{E}_\perp + \hat{E}_\perp^{cl}(\alpha,t)$$

We also have $|\psi'(t)\rangle = \hat{T}_\alpha|\alpha(t)\rangle = |0\rangle$

Action of the unitary transformation $\hat{T}_\alpha(t)$

$$\boxed{\begin{aligned} \hat{E}_\perp' &= \hat{T}_\alpha(t)\,\hat{E}_\perp\,\hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{cl}(\alpha,t) \\ |\psi'(t)\rangle &= T_\alpha(t)\,|\alpha(t)\rangle = |0\rangle \end{aligned}}$$

We can work with

$\hat{E}_\perp, |\alpha(t)\rangle$ or $\hat{E}_\perp + E_\perp^{cl}(\alpha,t), |0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!

Quantized Light – Matter Interactions

General Problem:

Quantum Atom $\longleftrightarrow$ **Quantized Modes Of EM Field**

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^{+}_{\vec{k}}\hat{a}_{\vec{k}} + \tfrac{1}{2}\right) \qquad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}}\cdot\hat{\vec{E}}(\vec{r},t) \qquad \text{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\hat{\sigma}_{ij} \qquad (2)$$

Field Operator:

$$\hat{\vec{E}}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.C., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right) \qquad (3)$$

Quantized Light – Matter Interactions

Dipole Operator:

(2) $$\boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij}}$$

Field Operator:

$$\hat{\vec{E}}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.C., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

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Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

(3) $$\boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij} \sum_{\vec{k}} -\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{\sigma}_{ij} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij} \sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

where $g^{(ij)}_{\vec{k}} = \dfrac{\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}}}{\hbar}$

Rabi Freq., note sign convention

2-level atom ⇨ $(i,j) = (1,2)$ **:**

$$\boxed{\hat{H}_{AF} = \hbar \sum_{\vec{k}} (g_{\vec{k}} \hat{\sigma}_{21} + g^{*}_{\vec{k}} \hat{\sigma}_{12}) (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Define:

$$\left.\begin{aligned} \hat{\sigma}_{+} &= \hat{\sigma}_{21} = |2\rangle\langle 1| \\ \hat{\sigma}_{-} &= \hat{\sigma}_{12} = |1\rangle\langle 2| \\ \hat{\sigma}_{z} &= \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1| \end{aligned}\right\} \Rightarrow$$

Pauli matrices

$$\hat{\sigma}_x = \tfrac{1}{2}(\hat{\sigma}_{+} + \hat{\sigma}_{-})$$

$$\hat{\sigma}_y = \tfrac{1}{2i}(\hat{\sigma}_{+} - \hat{\sigma}_{-})$$

$$\hat{\sigma}_z$$

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\boxed{\begin{aligned}\hat H_{AF} &= \sum_{ij}\sum_{\vec k} -\vec r_{ij}\cdot\vec\varepsilon_{\vec k}\,\mathcal{E}_{\vec k}\,\hat\sigma_{ij}\,(\hat a_{\vec k}+\hat a^+_{\vec k})\\ &= \sum_{ij}\sum_{\vec k}\hbar\, g^{(ij)}_{\vec k}(\hat a_{\vec k}+\hat a^+_{\vec k})\\ &\text{where}\quad g^{(ij)}_{\vec k} = \frac{\vec r_{ij}\cdot\vec\varepsilon_{\vec k}\,\mathcal{E}_{\vec k}}{\hbar}\end{aligned}}$$

Rabi Freq., note sign convention

2-level atom ⇨ $(i,j)=(1,2)$:

$$\boxed{\hat H_{AF} = \hbar\sum_{\vec k}(g_{\vec k}\hat\sigma_{21}+g^*_{\vec k}\hat\sigma_{12})(\hat a_{\vec k}+\hat a^+_{\vec k})}$$

Define:

$$\left.\begin{aligned}\hat\sigma_+ &= \hat\sigma_{21} = |2\rangle\langle 1|\\ \hat\sigma_- &= \hat\sigma_{12} = |1\rangle\langle 2|\\ \hat\sigma_z &= \hat\sigma_{22}-\hat\sigma_{11} = |2\rangle\langle 2| - |1\rangle\langle 1|\end{aligned}\right\}$$

Pauli matrices

$$\hat\sigma_x = \frac{1}{2}(\hat\sigma_+ + \hat\sigma_-)$$

$$\hat\sigma_y = \frac{1}{2i}(\hat\sigma_+ - \hat\sigma_-)$$

$$\hat\sigma_z$$

With this notation **(4)**

$$\hat H_{AF} = \hbar\sum_{\vec k}(g_{\vec k}\hat\sigma_+\hat a_{\vec k} + \cancel{g_{\vec k}\hat\sigma_+\hat a^+_{\vec k}} + \cancel{g_{\vec k}\hat\sigma_-\hat a_{\vec k}} + g_{\vec k}\hat\sigma_-\hat a^+_{\vec k})$$

Energy conservation?

$$\hat H_{AF} = \hbar\sum_{\vec k}(g_{\vec k}\hat\sigma_+\hat a_{\vec k} + g_{\vec k}\hat\sigma_-\hat a^+_{\vec k})$$

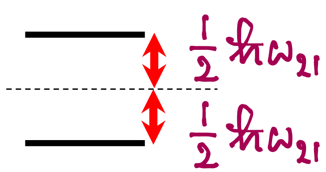
Putting it all together

$$\boxed{\begin{aligned}&\hat H = \hat H_F + \hat H_A + \hat H_{AF} = \qquad\qquad \textbf{(5)}\\ &\sum_{\vec k}\hbar\omega_{\vec k}\hat a^+_{\vec k}\hat a_{\vec k} + \frac{1}{2}\hbar\omega_{21}\hat\sigma_z + \sum_{\vec k}\hbar(g_{\vec k}\hat\sigma_+\hat a_{\vec k} + g_{\vec k}\hat\sigma_-\hat a^+_{\vec k})\end{aligned}}$$

We changed the zero point for energy by subtracting

$\sum_{\vec k}\frac{1}{2}\hbar\omega_{\vec k}$ and $\frac{1}{2}(E_2-E_1)$

field atom



Quantized Light – Matter Interactions

With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}^+_{\vec{k}}} + \cancel{g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

Putting it all together

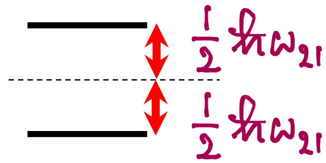
(5)

$$\boxed{\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}^+_{\vec{k}} + \frac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)}$$

Foundational result for remainder of course

We changed the zero point for energy by subtracting

$$\underbrace{\sum_{\vec{k}} \frac{1}{2} \hbar \omega_{\vec{k}}}_{\text{field}} \quad \text{and} \quad \underbrace{\frac{1}{2}(E_2 - E_1)}_{\text{atom}}$$



Quantized Light – Matter Interactions

With this notation

(4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}^+_{\vec{k}}} + \cancel{g_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

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Putting it all together

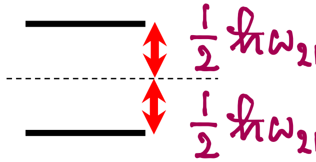
(5)

$$\boxed{\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)}$$

We changed the zero point for energy by subtracting

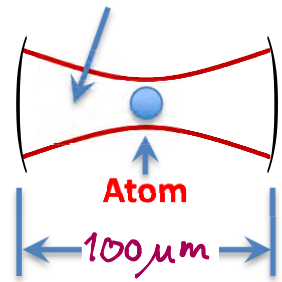
$\sum_{\vec{k}} \frac{1}{2}\hbar\omega_{\vec{k}}$ and $\frac{1}{2}(E_2 - E_1)$

field atom



Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

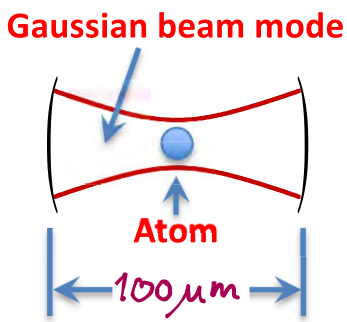
Single-mode (Jaynes-Cummings) Hamiltonian

$$\boxed{\hat{H} = \underbrace{\hbar\omega \hat{a}^+\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^+)}_{H_{AF}}}$$

Quantized Light–Matter Interactions

Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$$c/2L \gg A_{21}$$

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Single-mode (Jaynes-Cummings) Hamiltonian

$$\boxed{\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^{+})}_{H_{AF}}}$$

Paradigm for spin-1/2 coupled to QHO

- Atom in high-Q Cavity *)
- Quantum dot in high-Q Cavity
- Rydberg atom in superconducting μw Cavity
- Superconducting qubit in superconducting μw Cavity
- Superconducting qubit in superconducting μw stripline Cavity (Circuit QED)
- Trapped ion with quantized COM motion *)

*) Nobel Prize in Physics 2012