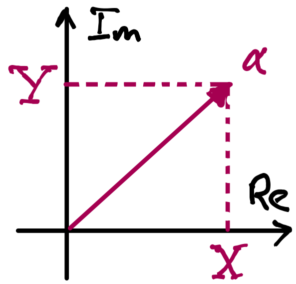
Quantum States of the Quantized Field

Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k \alpha_k(t)} e^{ikz} + c.c.$$

complex amplitude for mode e^{ikz}



Define

$$\boxed{\begin{aligned} X(t) &= Re[\alpha_k(t)] = \tfrac{1}{2}[\alpha_k(t) + \alpha_k^*(t)] = Q(t) \\ Y(t) &= Im[\alpha_k(t)] = \tfrac{1}{2i}[\alpha_k(t) - \alpha_k^*(t)] = P(t) \end{aligned}}$$

Quantization: $\alpha \to \hat{a}, \quad \alpha^* \to \hat{a}^+$

$$\left.\begin{aligned} \hat{X}(t) &= \tfrac{1}{2}[\hat{a}_k(t) + \hat{a}_k^+(t)] = \hat{Q}(t) \\ \hat{Y}(t) &= \tfrac{1}{2i}[\hat{a}_k(t) - \hat{a}_k^+(t)] = \hat{P}(t) \end{aligned}\right\} \quad [\hat{X}(t), \hat{Y}(t)] = i/2$$

$$\begin{aligned} \hat{E}(z,t) &= \mathcal{E}_k\left(\hat{X}(t) + i\hat{Y}(t)\right)e^{ikz} + H.C. \\ &= \mathcal{E}_k\left[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)\right] \end{aligned}$$

– same info, easier to work with –

Quantum States of the Field in Mode k

Number States (Foch states)

$$\boxed{\hat{a}^+\hat{a}|n\rangle = n|n\rangle}$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \tfrac{1}{2}(n + 1/2)$$

$$\boxed{\Delta X \Delta Y = \tfrac{1}{2}(n + 1/2)}$$

– HIGHLY non-classical, $\langle \hat{E} \rangle = 0$

– VERY hard to make for large n

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Coherent States (Quasi-classical states)

– Closest approximation to classical field

– See Cohen-Tannoudj, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle\hat{X}(t)\rangle = \langle\psi|\hat{X}(t)|\psi\rangle = X(t),\quad \langle\hat{Y}(t)\rangle = Y(t)$$

$$\langle\hat{H}(t)\rangle = \hbar\omega\left(|\alpha(t)|^2 + 1/2\right)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}$$

equivalently

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(1) $\langle\hat{a}(0)\rangle = \langle\psi|\hat{a}(0)|\psi\rangle = \alpha(0)$

(2) $\langle\hat{a}^\dagger(0)\hat{a}(0)\rangle = \alpha(0)^*\alpha(0)$

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Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

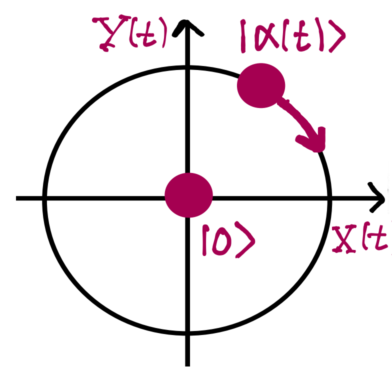
Physical properties

$$\langle \hat{X}(t)\rangle = \mathrm{Re}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\langle \hat{Y}(t)\rangle = \mathrm{Im}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$

$$\Delta X\,\Delta Y = 1/4$$



Quantum States of the Quantized Field

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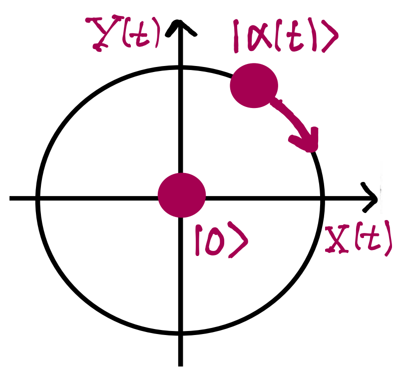

Photon statistics

Measure $\hat{N}$ ⇒ outcomes n; $P(n) = \langle\alpha|n\rangle\langle n|\alpha\rangle = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$

Poisson distribution w/ mean $\bar{n} = |\alpha|^2$; variance $\Delta n^2 = |\alpha|^2$

$$\boxed{\Delta n = \sqrt{\bar{n}}} \quad \text{– Shot Noise}$$

Quantum States of the Quantized Field

Photon statistics

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⇓

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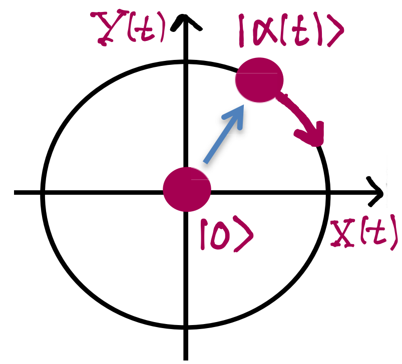
⇓

$\boxed{\Delta n = \sqrt{\bar{n}}}$ **– Shot Noise**

More about Coherent States



⇒ **Coherent States as translated Vacuum States?**

Generating Coherent States from the Vacuum

Definition: $\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$

Unitary, equals translation

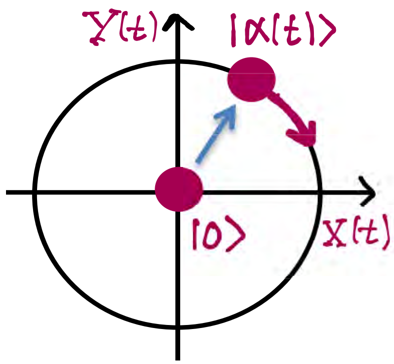
Glaubers formula (from BCH formula)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Quantum States of the Quantized Field

More about Coherent States



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Apply to $[\underbrace{\alpha \hat{a}^{+}}_{\hat{A}}, \underbrace{-\alpha^{*} \hat{a}}_{\hat{B}}] = \underbrace{\alpha^{*}\alpha}_{[\hat{A},\hat{B}]}$

$$\boxed{\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^{+}} e^{-\alpha \hat{a}}}$$

Remember: $\hat{a}|0\rangle = 0$

$$e^{-\alpha^{*}\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^{*}\hat{a})^n}{n!}|0\rangle = |0\rangle$$

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^{+}}|0\rangle$$

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Quantum States of the Quantized Field

Apply to $[\alpha\hat{a}^{+}, -\alpha^{*}\hat{a}] = \alpha^{*}\alpha$

$$\underset{\hat{A}}{\alpha\hat{a}^{+}},\quad \underset{\hat{B}}{-\alpha^{*}\hat{a}},\quad \underset{[\hat{A},\hat{B}]}{\alpha^{*}\alpha}$$

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$$\boxed{\hat{D}(\alpha)|0\rangle = |\alpha\rangle}$$

OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat{a}^{+} - \alpha^{*}\hat{a} = (\alpha-\alpha^{*})\hat{X} + i(\alpha+\alpha^{*})\hat{Y}$$
$$= i2Y\hat{X} + i2X\hat{Y}$$

where $X = \langle\alpha|\hat{X}|\alpha\rangle$, $Y = \langle\alpha|\hat{Y}|\alpha\rangle$

Glaubers formula again:

$$\boxed{\hat{D}(\alpha) = e^{i2Y\hat{X} + i2X\hat{Y}} = e^{-XY/4} e^{i2Y\hat{X}} e^{i2X\hat{Y}}}$$

Recall: $\hat{S}(q) = e^{-iq\hat{p}/\hbar}$ **translation by** q

$\hat{S}(p) = e^{-ip\hat{q}/\hbar}$ **translation by** p

where $q = q_0 X$, $p = p_0 Y$ & $x_0 p_0 = 2\hbar$

$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$

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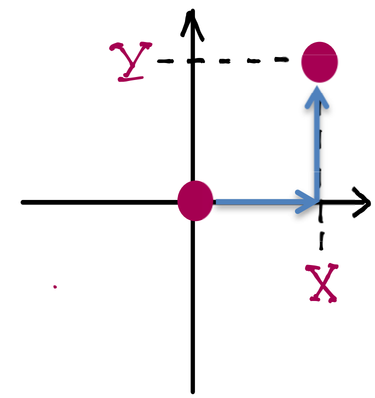
$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$

This gives us

$$\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}$$

$\hat{D}(\alpha)$ translates along X then Y



Discussion –
How to do this?