Problem I

(a) We can relate the transmission to the imaginary index of refraction,

$$T = e^{-aL} \quad \Rightarrow \quad a = \frac{-\ln(T)}{L} = \frac{2n_{\mathrm{I}}(\omega_{\mathrm{Li}})\omega_{\mathrm{Li}}}{c} \quad \Rightarrow \quad n_{\mathrm{I}}(\omega_{\mathrm{Li}}) = \frac{-\ln(T)c}{2\omega_{\mathrm{Li}}L}$$

Then, using the electron oscillator model in the near-resonance, weakly polarizable limit, and setting $\Delta = 0$, we have

$$n_{\mathrm{I}}(\Delta = 0) = \frac{Ne^2}{4\varepsilon_0 m\omega_{\mathrm{Li}}}\frac{\beta}{\Delta^2+\beta^2} = \frac{Ne^2}{4\varepsilon_0 m\omega_{\mathrm{Li}}\beta} = \frac{-\ln(T)c}{2\omega_{\mathrm{Li}}L} \Rightarrow N = \frac{-2\varepsilon_0 m\beta\ln(T)c}{Le^2}$$

$$= \frac{-2\times 8.854\times 10^{-12}\,\tfrac{\mathrm{F}}{\mathrm{m}}\times 9.11\times 10^{-31}\mathrm{kg}\times 3.68\times 10^{7}\,\tfrac{1}{\mathrm{s}}\times \ln(0.001)\times 3.00\times 10^{8}\,\tfrac{\mathrm{m}}{\mathrm{s}}}{0.001\mathrm{m}\times(1.602\times 10^{-19}\mathrm{C})^2}$$

$$= 4.79\times 10^{16}\,\mathrm{m}^{-3} \quad \text{(units check out)}$$

(b) We have $n_{\mathrm{I}}(\Delta) = \dfrac{Ne^2}{4\varepsilon_0 m\omega_{\mathrm{Li}}}\dfrac{\beta}{\Delta^2+\beta^2} \Rightarrow n_{\mathrm{R}}(\Delta = 0) = 1 + \dfrac{Ne^2}{4\varepsilon_0 m\omega_{\mathrm{Li}}}\dfrac{\Delta}{\Delta^2+\beta^2} = 1 + \dfrac{\Delta}{\beta}n_{\mathrm{I}}(\Delta)$

(c) The maximum real index of refraction occurs when

$$\frac{d}{d\Delta}\frac{\Delta}{\Delta^2+\beta^2} = \frac{1}{\Delta^2+\beta^2} - \frac{2\Delta^2}{(\Delta^2+\beta^2)^2} = 0 \Rightarrow \Delta^2 = \beta^2 \Rightarrow \Delta = \pm\beta$$

Picking $\Delta = \beta$ where the real index of refraction is greater than 1, we have

$$n_{\mathrm{R}}(\Delta = \beta) \;=\; 1 + \frac{\Delta}{\beta}n_{\mathrm{R}} \;=\; 1 + n_{\mathrm{I}}(\Delta = 0) \;=\; 1 - \frac{\ln(T)c}{2\omega L}$$

$$=\; 1 - \frac{\ln(0.001)\times 3\times 10^{8}}{2\times 3.10\times 10^{15}\times 0.001} = 1.000334$$

Problem II

(a) For the density operator, the Schrödinger equation in the RWA takes the form

$i\hbar\dot{\rho} = [H,\rho]$, or for the individual matrix elements, $\dot{\rho}_{ij} = \sum_k H_{ik}\rho_{kj} - \rho_{ik}H_{kj}$

(b) There are 3 populations and 6 coherences. Borrowing from (a) above and noting that some terms are zero or cancel out, we get

$$\dot{\rho}_{11} = -\frac{i}{h}(H_{11}\rho_{11} + H_{12}\rho_{21} + H_{13}\rho_{31} - \rho_{11}H_{11} - \rho_{12}H_{21} - \rho_{13}H_{31}) = -\frac{i\chi}{2}(\rho_{21} - \rho_{12})$$

$$\dot{\rho}_{22} = -\frac{i}{h}(H_{21}\rho_{12} + H_{22}\rho_{22} + H_{23}\rho_{32} - \rho_{21}H_{12} - \rho_{22}H_{22} - \rho_{23}H_{32}) = -\frac{i\chi}{2}(\rho_{12} - \rho_{21} + \rho_{32} - \rho_{23})$$

$$\dot{\rho}_{33} = -\frac{i}{h}(H_{31}\rho_{13} + H_{32}\rho_{23} + H_{33}\rho_{33} - \rho_{31}H_{13} - \rho_{32}H_{23} - \rho_{33}H_{33}) = -\frac{i\chi}{2}(\rho_{23} - \rho_{32})$$

$$\dot{\rho}_{12} = -\frac{i}{h}(H_{11}\rho_{12} + H_{12}\rho_{22} + H_{13}\rho_{32} - \rho_{11}H_{12} - \rho_{12}H_{22} - \rho_{13}H_{32}) = i\Delta\rho_{12} - \frac{i\chi}{2}(\rho_{22} - \rho_{11} - \rho_{13}) = \dot{\rho}_{12}^*$$

$$\dot{\rho}_{32} = -\frac{i}{h}(H_{31}\rho_{12} + H_{32}\rho_{22} + H_{33}\rho_{32} - \rho_{31}H_{12} - \rho_{32}H_{22} - \rho_{33}H_{32}) = i\Delta\rho_{32} - \frac{i\chi}{2}(\rho_{22} - \rho_{33} - \rho_{31}) = \dot{\rho}_{32}^*$$

$$\dot{\rho}_{13} = -\frac{i}{h}(H_{12}\rho_{23} - \rho_{12}H_{23}) = -\frac{i\chi}{2}(\rho_{23} - \rho_{12}) = \dot{\rho}_{31}^*$$

Note that $\dot{\rho}_{11} + \dot{\rho}_{22} + \dot{\rho}_{33}$ as required. Also, the first three equations are what one expects when having two 2-level systems that share an excited state. The same is true for equations four and five, except for the appearance of coherences ρ_{13} and ρ_{31}. The is no immediately intuitive way to check that equation six is correct.

(c) Decay from the excited state occurs at rate A, and feeds the populations in each of the ground state at rates $A/2$. Furthermore, we know from our conceptual analysis of spontaneous decay in the two-level system that coherences decay at half the rate at which atoms disappear from the excited state, i. e., at rates $A/2$. The coherence between the ground states is not affected, on average, by the probability amplitudes that arrive from the excited state because of their randomly fluctuating phase. Thus

$$\dot{\rho}_{11} = \frac{A}{2}\rho_{22} - \frac{i\chi}{2}(\rho_{21} - \rho_{12}),$$

$$\dot{\rho}_{22} = -\frac{A}{2}\rho_{22} - \frac{i\chi}{2}(\rho_{12} - \rho_{21} + \rho_{32} - \rho_{23})$$

$$\dot{\rho}_{33} = \frac{A}{2}\rho_{22} - \frac{i\chi}{2}(\rho_{23} - \rho_{32})$$

$$\dot{\rho}_{12} = (i\Delta - A/2)\rho_{12} - \frac{i\chi}{2}(\rho_{22} - \rho_{11}) = \dot{\rho}_{12}^*,$$

$$\dot{\rho}_{32} = (i\Delta - A/2)\rho_{32} - \frac{i\chi}{2}(\rho_{22} - \rho_{33}) = \dot{\rho}_{32}^*$$

$$\dot{\rho}_{13} = -\frac{i}{h}(H_{12}\rho_{23} - \rho_{12}H_{23}) = -\frac{i\chi}{2}(\rho_{23} - \rho_{12}) = \dot{\rho}_{31}^*$$