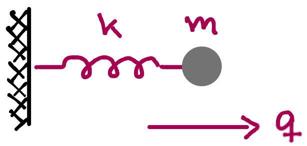
Quantum Electrodynamics – Intro to Field Theory

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



$\omega = \sqrt{k/m}$

osc. freq.

dynamical coordinate.

Kinetic Energy: $T = \frac{1}{2} m\dot{q}^2$

Potential Energy: $V = \frac{1}{2} k q^2 = \frac{1}{2} m\omega^2 q^2$

<u>Lagrangian</u>: $\mathcal{L} = T - V = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \;\Rightarrow\; \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

Conjugate momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

$$\left.\begin{array}{l} \dot{q} = \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \end{array}\right\} \Rightarrow \ddot{q} + \omega^2 q = 0$$

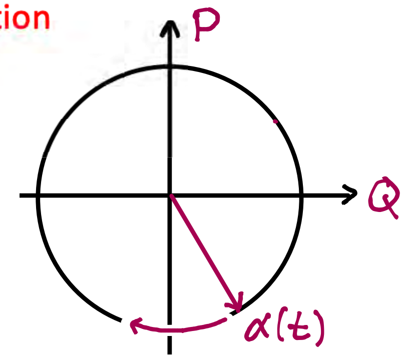
solution

Phase plane

Scaled variables

$Q \equiv q/q_0, \quad P = p/p_0$

$$\alpha = Q + iP \quad \left\{ \begin{array}{l} Q = \mathrm{Re}[\alpha] \\ P = \mathrm{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{array}\right.$$



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Conjugate momentum

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Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$\boxed{q \to \hat{q}, \quad p \to \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar}$$

Choose $E_0 = \hbar\omega \Rightarrow q_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \to \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right)$$

$$[\hat{a}, \hat{a}^+] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^+\hat{a} + 1/2)$$

$\hat{N} = \hat{a}^+\hat{a}$ **(number operator)**

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Quantum Harmonic Oscillator

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$\hat{N} = \hat{a}^+\hat{a}$ (number operator)

Commutator $[\hat{H}, \hat{N}] = 0$

⇒ joint energy/number states $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\left.\begin{aligned}[\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a}\end{aligned}\right\} \Rightarrow \begin{aligned}\hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}|0\rangle &= 0\end{aligned}$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle$$

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Commutator $[\hat{H}, \hat{N}] = 0$

➡ **joint energy/number states** $|n\rangle$

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Commutators

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Generating excited states

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle}$$

Expectation values for $\hat{q}$ and $\hat{p}$ in number states

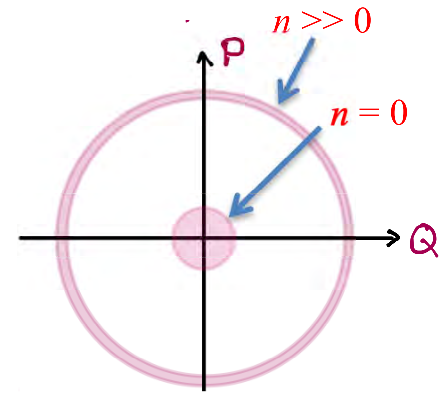
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n+1/2) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n+1/2) \neq 0$$

$$\Delta q\,\Delta p = \frac{q_0 p_0}{2}(n+1/2) = \hbar(n+1/2)$$

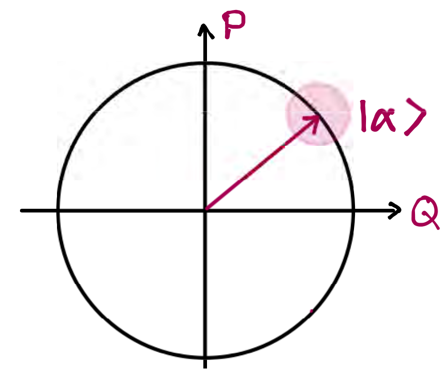
Phase space visualization of number states



Quasi-classical (coherent) state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_i \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\Delta q\,\Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



End 03-28-2022

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03-30-2022

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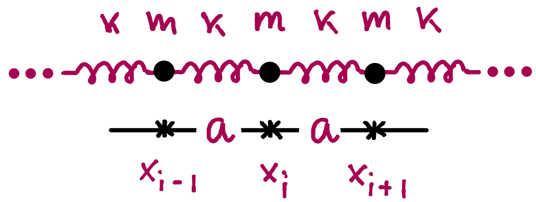
Lagrange formulation of 1D Scalar Field

"vibronium"



Configuration space = $\{x_i\}$ **(set of *N* osc. positions)**

$$T = \sum_{i=1} \tfrac{1}{2} m \dot{x}_i^2, \quad V = \sum_{i=1}^{N} \tfrac{1}{2} \kappa (x_{i+1} - x_i)^2$$

Lagrangian, equations of motion

Continuum limit $\Rightarrow$ **Elastic rod**

$N \to \infty$ $\quad m/a \to \mu$ $\leftarrow$ linear mass density

$a \to dx$ $\quad \kappa a \to Y$ $\leftarrow$ Youngs modulus

$\{x_i\} \to y(x)$ $\leftarrow$ displacement field (sound)

Rewrite

$$\boxed{\begin{aligned} T &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^2 = \int dx\, \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \\ V &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2 = \int dx\, \tfrac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2 \end{aligned}}$$

Lagrangian:

$$\boxed{\mathcal{L} = T - V = \int dx\, \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \int dx\, \tfrac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2}$$

Notes, Homework $\Rightarrow$ **Scalar wave equation**

$$\boxed{\frac{\partial^2 y}{\partial t^2} - \frac{Y}{\mu}\frac{\partial y^2}{\partial x^2} = 0}$$

– Not yet ready for Quantization –

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Rewrite

$$T = \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^{\,2} = \int dx \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

$$V = \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2 = \int dx \tfrac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \int dx \tfrac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2$$

Notes, Homework ⇨ **Scalar wave equation**

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Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t}u(x)$ ⇨

$$\ddot{y} - v^2 y'' = -\omega^2 g(t)u(x) - v^2 g(t)u''(x) = 0$$

$$u''(x) = -k^2 u(x), \quad k = \omega/v$$

Solutions in cavity:

$$u_k(x) = \sqrt{\frac{2}{L}}\sin(kx), \quad k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

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Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

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Solutions in cavity:

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These standing waves are a set of Normal Modes

These modes are orthonormal and complete

$$\boxed{y(x,t) = \sqrt{L}\sum_k q_k(t)\, u_k(x)}$$

Normal mode expansion of $y(x,t)$ **in basis** $u_k(x)$

Lagrangian for the acoustic field:

$$T = \int dx\, \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \sum_{k,k'} \frac{1}{2}\underbrace{\mu L}_{M}\, \dot{q}_k \dot{q}_{k'} \underbrace{\int dx\, u_k(x) u_{k'}(x)}_{\delta_{kk'}}$$

$$= \sum_k \frac{1}{2} M \dot{q}_k^2$$

$$V = \int dx\, \frac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2 = \sum_{kk'} \frac{1}{2} Y L\, q_k q_{k'} \int dx \left(\frac{\partial u_k}{\partial x}\right)\left(\frac{\partial u_{k'}}{\partial x}\right)$$

$$= \sum_k \frac{1}{2} M \omega_k^2 q_k^2$$

$$\boxed{\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 - \frac{1}{2} M \omega_k^2 q_k^2\right) = \sum_k \mathcal{L}_k}$$

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The rest now follows from the Lagrangian

$$\mathcal{L} = T - V = \sum_k \left(\tfrac{1}{2} M \dot{q}_k^2 - \tfrac{1}{2} M \omega_k^2 q_k^2 \right) = \sum_k \mathcal{L}_k$$

Canonical Momentum

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = M \dot{q}_k$$

Hamiltonian

$$\mathcal{H}(\{p_k, q_k\}) = T + V = \sum_k \left(\frac{p_k^2}{2M} + \tfrac{1}{2} M \omega_k^2 q_k^2 \right)$$

(collection of SHO's, one for each normal mode)

Following the standard recipe…

$E_{0,k} = \hbar\omega_k$, $\quad q_{0,k} = \sqrt{2\hbar / M\omega_k}$, $\quad p_{0,k} = \sqrt{2M\hbar\omega_k}$

$Q_k = q_k / q_{0,k}$, $\quad P_k = p_k / p_{0,k}$, $\quad \alpha_k = Q_k + i P_k$

… we get solutions

$$\alpha_k(t) = Q_k(t) + i P_k(t) = \alpha_k(0)\, e^{-i\omega_k t}$$

This finally gives us

$$\mathcal{H} = \sum_k \hbar\omega_k \left(Q_k^2 + P_k^2 \right) = \sum_k \hbar\omega_k \alpha_k^* \alpha_k$$

$$y(x,t) = \sqrt{L} \sum_k q_k(t)\, u_k(x)$$

$$= \frac{1}{2} \sum_k \sqrt{L q_{0,k}^2} \left(\alpha_k(t)\, u_k(x) + \alpha_k^*(t)\, u_k^*(x) \right)$$

allows real or complex $u_k(x)$

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… we get solutions

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allows real or complex $u_k(x)$

Formal Quantization Procedure:

$$q_k \to \hat{q}_k, \quad p_k \to \hat{p}_k, \quad \alpha_k \to \hat{a}_k$$

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar\,\delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}] = 0$$

Note: $k \neq k'$ ➡ **operators commute**

(normal modes = independent degs. of freedom)

Hamiltonian & Quantized fields

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + 1/2\right)$$

$$\hat{y}(x) = \sqrt{L}\sum_k \hat{q}_k\, u_k(x) = \sum_k \sqrt{L q_{0,k}^2}\left(\hat{a}_k u_k(x) + \hat{a}_k^\dagger u_k^*(x)\right)$$

$$\hat{\Pi}(x) = \frac{1}{\sqrt{L}}\sum_k \hat{p}_k u_k(x) = -i\sum_k \sqrt{\frac{p_{0,k}^2}{L}}\left(\hat{a}_k u_k(x) - \hat{a}_k^\dagger u_k^*(x)\right)$$

field $\hat{y}(x)$ **and canonical momentum field** $\hat{\Pi}(x)$

➡ $\left[\hat{y}(x), \hat{\Pi}(x')\right] = i\hbar\,\delta(x-x')$

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Formal Quantization Procedure:

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$$\hat{\pi}(x) = \frac{1}{\sqrt{L}}\sum_k \hat{p}_k\, u_k(x) = -i\sum_k \sqrt{\frac{p_{0k}^2}{L}}\left(\hat{a}_k u_k(x) - \hat{a}_k^\dagger u_k^*(x)\right)$$

field $\hat{y}(x)$ and canonical momentum field $\hat{\pi}(x)$

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Quantum States:

The quantum field is a collection of QSHO's

State space = tensor product of SHO spaces

SHO space

Fock Space $\mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \dots , \mathcal{E}_{k_j}$

SHO state

Fock State $|\{n_{k_1}, n_{k_2}, \dots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots |n_{k_j}\rangle$

$\hat{a}_{k_i}, \hat{a}_{k_i}^\dagger$ destroy/create excitations in mode k_i

Vacuum State $|0\rangle$ ← zero quanta in every mode

Favorite Question: **What is a Phonon?**

End 03-30-2022