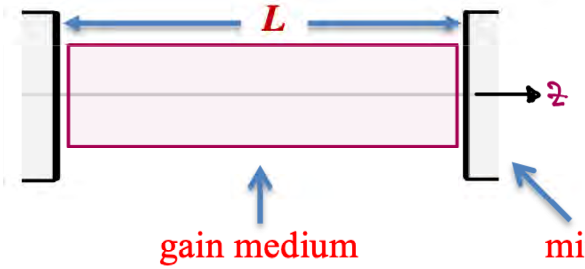
Semi-Classical Laser Theory

Lasing action: requires a gain medium & feedback



(∗) As usual we simplify to focus on the key concepts ⇨ 1D cavity

(∗) For spherical mirror resonators, see M&E Ch. 14

Optical Resonator/Cavity ⇨

Eigenmodes of the Electromagnetic Field

Plane Parallel Mirrors ⇨ **standing waves**

Length L ⇨ wave number for m'th mode

$$k = \frac{m\pi}{L}, \quad m \text{ integer}$$

Field in the m'th mode

$$\boxed{\vec{E}_m(z,t) = \vec{\varepsilon}_m \mathcal{E}_m(t) \sin(k_m z) e^{-i\omega t}}$$

Slowly Varying Envelope

Note: Gain ⇨ dispersion in cavity

$$\omega \neq k_m c = \omega_m$$

Laser freq. Vacuum mode freq.

Polarization density, m'th mode

$$\vec{P}_m(z,t) = \vec{\varepsilon}_m 2N\mu^* \rho_{21}^{(m)}(z,t) \sin(k_m z) e^{-i\omega t}$$

$$\vec{P}(z,t) = \sum_m \vec{P}_m(z,t)$$

Total polarization density in all modes

Note: Saturation effects ⇨ Mode cross-talk

Semi-Classical Laser Theory

Field in the $m'th$ mode

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Wave eq. in a Resonator

– mimic loss by including current $\vec{J} = \sigma\vec{E}$

finite conductivity

4th Maxwell Eq.: $\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \sigma\vec{E}$

Wave Eq. in resonator, with distributed loss

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\kappa}{c^2}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}$$

$$\kappa = \sigma/\varepsilon_0$$

Phenomenological loss constant (losses + output coupling)

units 1/s

Wave Eq. for $m'th$ mode in the resonator

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\kappa}{c^2}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{\varepsilon}_m \mathcal{E}_m(t)\sin(k_m z)e^{-i\omega t}$$

$$= \frac{\vec{\varepsilon}_m}{\varepsilon_0 c^2} 2N\mu^* \sin(k_m z)\frac{\partial^2}{\partial t^2}\left(\rho_{21}^{(m)}(t)e^{-i\omega t}\right)$$

Semi-Classical Laser Theory

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$\kappa = \sigma/\varepsilon_0$ ← Phenomenological loss constant (losses + output coupling)

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Wave Eq. for m'th mode in the resonator

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$$= \frac{\vec{\varepsilon}_m}{\varepsilon_0 c^2} 2N\mu^* \sin(k_m z)\frac{\partial^2}{\partial t^2}\left(\rho_{21}^{(m)}(t)e^{-i\omega t}\right)$$

Apply SVEA & resonant approx., $\omega - \omega_m \ll \omega$ (HW)

$$\left[-i(\omega-\omega_m) + \frac{\kappa}{2} + \frac{\partial}{\partial t}\right]\mathcal{E}_m(t) = \frac{i\omega}{\varepsilon_0} N\mu^* \rho_{21}^{(m)}(t)$$

Quasi-steady state solution:

Fast atomic response
High-Q cavity
$(\beta \gg \kappa)$
} ⇒ $\rho_{21}^{(m)}(t)$ in S. S. given $\mathcal{E}(t)$

We can adiabatically eliminate $\rho_{21}^{(m)}(t)$ by replacing w/ S.S value given $\mathcal{E}(t)$, ρ_{11} & ρ_{22}

$$\rho_{21}(t) = -\frac{i\mu\mathcal{E}(t)}{2\hbar}\frac{\beta - i\Delta}{\beta^2+\Delta^2}(\rho_{22}-\rho_{11})$$

Note: We will allow for some external process that potentially creates a population inversion

Semi-Classical Laser Theory

Apply SVEA **& resonant approx.,** $\omega - \omega_m \ll \omega$ **(HW)**

$$\left[-i(\omega-\omega_m)+\frac{\kappa}{2}+\frac{\partial}{\partial t}\right]\mathcal{E}_m(t)=\frac{i\omega}{\varepsilon_0}N\mu^*\rho_{21}^{(m)}(t)$$

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Substitute in Equation for $\mathcal{E}_m(t)$

$$\left[-i(\omega-\omega_m)+\frac{\kappa}{2}+\frac{\partial}{\partial t}\right]\mathcal{E}_m(t) = \frac{N|\mu|^2\omega}{2\varepsilon_0\hbar}\,\frac{\beta-i\Delta}{\beta^2+\Delta^2}(\rho_{22}-\rho_{11})\mathcal{E}_m(t)$$

Let $N_1 = N\rho_{11}$, $N_2 = N\rho_{22}$ **and define**

$$g \equiv \frac{|\mu|^2\omega}{\varepsilon_0\hbar c}\,\frac{\beta}{\Delta^2+\beta^2}(N_2-N_1) = \sigma(\Delta)(N_2-N_1)$$

gain

$$\delta \equiv \frac{\Delta}{\beta}g = \frac{\Delta}{\beta}\sigma(\Delta)(N_2-N_1)$$

dispersion

Fundamental Eq. of Semiclassical Laser Theory

$$\boxed{\frac{\partial}{\partial t}\mathcal{E}_m(t)=\frac{1}{2}\left[-\kappa+2i(\omega-\omega_m)+c(g-i\delta)\right]\mathcal{E}_m(t)}$$

Semi-Classical Laser Theory

Substitute in Equation for $\mathcal{E}_m(t)$

$$\left[-i(\omega-\omega_m)+\frac{\kappa}{2}+\frac{\partial}{\partial t}\right]\mathcal{E}_m(t)$$
$$=\frac{N|\mu|^2\omega}{2\varepsilon_0\hbar}\frac{\beta-i\Delta}{\beta^2+\Delta^2}(\rho_{22}-\rho_{11})\mathcal{E}_m(t)$$

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The FESLT gives us insight into

- (∗) **Threshold behavior**
- (∗) **Laser intensity and power output**
- (∗) **Laser frequency and linewidth**

Equation for Laser intensity $I\propto\mathcal{E}^*\mathcal{E}$

$$\frac{dI}{dt}\propto\frac{\partial\mathcal{E}_m^*(t)}{dt}\mathcal{E}_m(t)+C.C.$$
$$=\frac{1}{2}\left[-\kappa-2i(\omega-\omega_m)+c(g-i\delta)\right]|\mathcal{E}_m(t)|^2+C.C.$$

$$\frac{dI}{dt}=(cg-\kappa)I$$

- $g>g_t$ **: exponential growth**
- $g<g_t$ **: exponential decay**

We define $g_t=\sigma(\Delta)\Delta N_t$, $\Delta N_t=\frac{\kappa}{c\sigma(\Delta)}$

Semi-Classical Laser Theory

The FESLT gives us insight into

- (∗) Threshold behavior
- (∗) Laser intensity and power output
- (∗) Laser frequency and linewidth

Equation for Laser intensity $I \propto \mathcal{E}^* \mathcal{E}$ ➡

$$\frac{dI}{dt} \propto \frac{\partial \mathcal{E}_m^*(t)}{dt} \mathcal{E}_m(t) + C.C.$$

$$= \frac{1}{2}\left[-\kappa - 2i(\omega - \omega_m) + C(g - i\delta)\right] |\mathcal{E}_m(t)|^2 + C.C.$$

$$\frac{dI}{dt} = (cg - \kappa) I$$ ➡

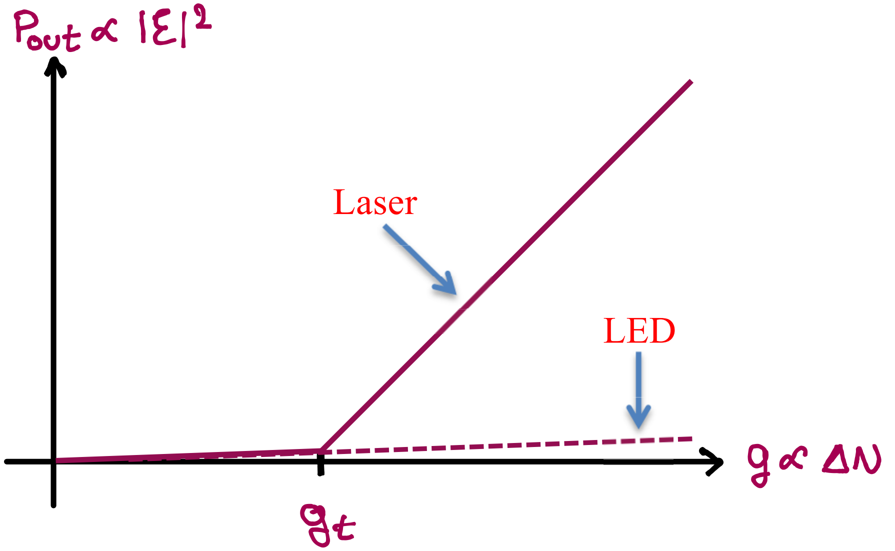
- $g > g_t$: exponential growth
- $g < g_t$: exponential decay

We define $g_t = \sigma(\Delta)\Delta N_t$, $\Delta N_t = \frac{\kappa}{c\sigma(\Delta)}$

These are Key parameters that characterizes a laser

$g_t = \frac{\kappa}{c}$	Threshold Gain
$\Delta N_t = \frac{\kappa}{c\sigma(\Delta)}$	Threshold Inversion

Example: Diode lasers & threshold behavior



End 03-23-2022

Semi-Classical Laser Theory

Fundamental Eq. of Semiclassical Laser Theory

$$\frac{\partial \mathcal{E}_m(t)}{\partial t} = \frac{1}{2}\left[-\kappa + 2i(\omega-\omega_m) + c(g - i\delta)\right]\mathcal{E}_m(t)$$

$$g \equiv \sigma(\Delta)(N_2 - N_1) \qquad \delta \equiv \frac{\Delta}{\beta} g = \frac{\Delta}{\beta}\sigma(\Delta)(N_2-N_1)$$

gain dispersion

Laser Frequency in Steady State

Let $\frac{\partial}{\partial t}\mathcal{E}_m = 0$ and consider imaginary part of FESLT

$$i(\omega-\omega_m) - i\frac{c\delta}{2} = 0, \quad \text{with } \delta = \frac{\Delta}{\beta} g$$

$$\omega_m - \omega = -\frac{gc}{2\beta}\Delta = \frac{gc}{2\beta}(\omega - \omega_{21})$$

Solve for ω :

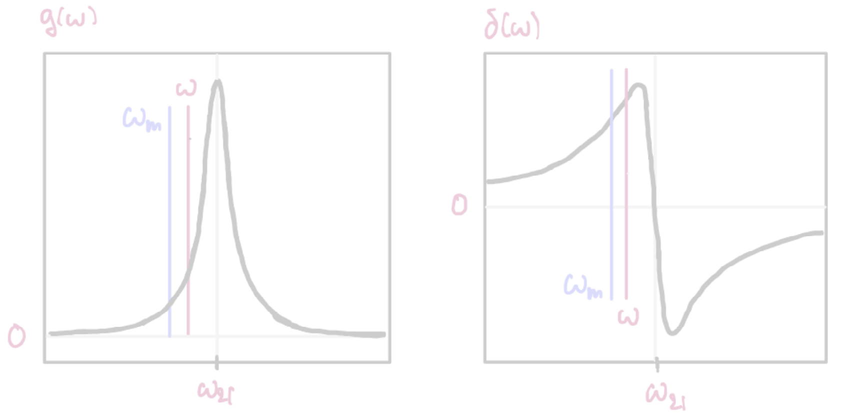
$$\omega = \frac{\omega_m + gc/_{2\beta}\,\omega_{21}}{1 + gc/2\beta} \approx \omega_m + \frac{cg}{2\beta}(\omega_{21} - \omega_m)$$

laser frequency

for $\frac{gc}{2\beta} \ll 1$

frequency pulling

Physical interpretation – note $\delta(\omega) > 0$ for $\omega < \omega_{21}$



From MBE's $n_R = 1 - \frac{\delta\omega}{2k}$ ⇨ $n_R < 1$

⇨ Optical L < physical L ⇨ ω increases

– Laser frequency is pulled towards resonance

Semi-Classical Laser Theory

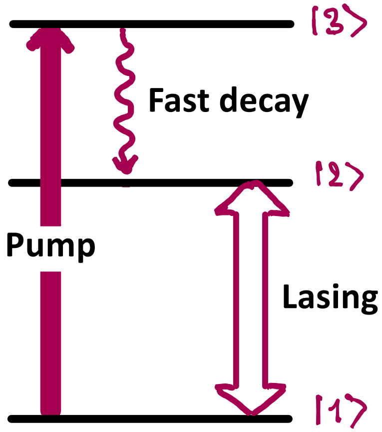
Gain requires Population Inversion ➡

Laser Pumping Schemes

3-Level System

Ruby Laser

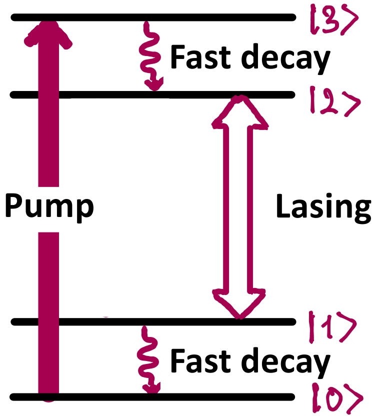
Hard to Pump!



4-Level System

Nd-YAG
Ti-Sappire
Er-Fiber (glass)
Organic Dye
Helium-Neon
Semiconductor

Easy to Pump!



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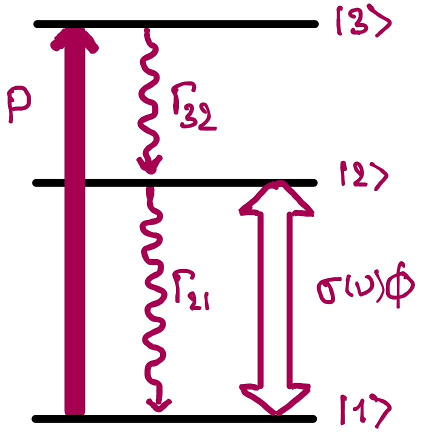
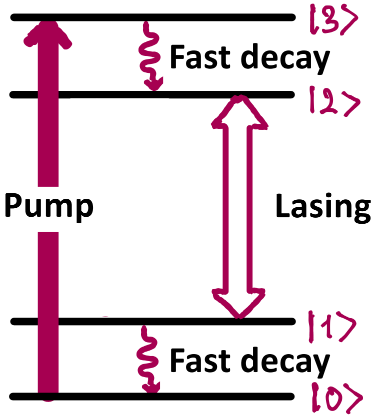


4-Level System

Nd-YAG
Ti-Sappire
Er-Fiber (glass)
Organic Dye
Helium-Neon
Semiconductor

Easy to Pump!



Population Rate Equations – 3 level System

Let $\Gamma_{32} \gg P,\ \Gamma_{21},\ \sigma(\nu)\phi$ ⇨ $N_3 \sim 0$

⇩

$$\boxed{\begin{aligned} \dot{N}_1 &= -PN_1 + \Gamma_{21}N_2 + \sigma(\nu)\phi(N_2 - N_1) \\ \dot{N}_2 &= PN_1 - \Gamma_{21}N_2 - \sigma(\nu)\phi(N_2 - N_1) \end{aligned}}$$

Steady State Solution (Home Work Set 6)

$$N_2 - N_1 = \frac{(P - \Gamma_{21})(N_2 + N_1)}{P + \Gamma_{21} + 2\sigma(\nu)\phi}$$

Use $\left\{ \begin{aligned} & N_1 + N_2 = N \\ & g(\nu) = \sigma(\nu)(N_2 - N_1) \end{aligned} \right.$ ⇨

$$g(\nu) = \sigma(\nu)\frac{(P - \Gamma_{21})N}{P + \Gamma_{21} + 2\sigma(\nu)\phi} > 0 \quad \text{iff} \quad P > \Gamma_{21}$$

Semi-Classical Laser Theory

Population Rate Equations – 3 level System

Let $\Gamma_{32} >> P, \ \Gamma_{21}, \ \sigma(\nu)\phi$ ⇨ $N_3 \sim 0$

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Divide top & bottom w/ $P + \Gamma_{21}$

$$g(\nu) = \frac{g_0(\nu)}{1 + \phi/\phi_{sat}}$$

Saturated Gain

$$g_0(\nu) = \sigma(\nu)\frac{(P - \Gamma_{21})N}{P + \Gamma_{21}}$$

Small Signal Gain

$$\phi_{sat} = \frac{P + \Gamma_{21}}{2\sigma(\nu)}$$

Saturation Flux

$$I_{sat} = h\nu\phi_{sat}$$

Saturation Intensity

Semi-Classical Laser Theory

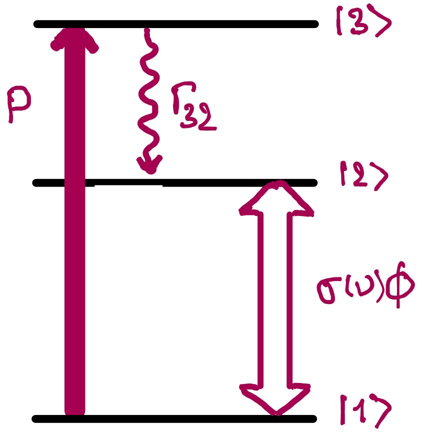
Gain requires Population Inversion ➡

Laser Pumping Schemes

3-Level System

Ruby Laser

Hard to Pump!



4-Level System

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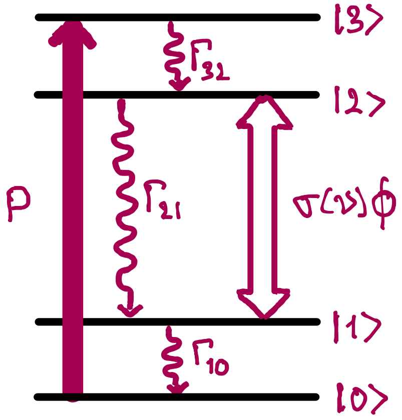
Semiconductor

Easy to Pump!



Population Rate Equations – 4 level System

Let $\Gamma_{32} \gg P,\ \Gamma_{21},\ \sigma(\nu)\phi$ ➡ $N_3 \sim 0$

$$\boxed{\begin{aligned}\dot{N}_0 &= -PN_0 + \Gamma_{10}N_1\\ \dot{N}_1 &= -\Gamma_{10}N_1 + \Gamma_{21}N_2 + \sigma(\nu)\phi(N_2 - N_1)\\ \dot{N}_2 &= PN_0 - \Gamma_{21}N_2 - \sigma(\nu)\phi(N_2 - N_1)\end{aligned}}$$

Steady State Solution (Home Work Set 6)

$$N_2 - N_1 = \frac{P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21} + (2P + \Gamma_{10})\sigma\phi}$$

Semi-Classical Laser Theory

Population Rate Equations – 4 level System

Let $\Gamma_{32} >> P,\ \Gamma_{21},\ \sigma(\nu)\phi$ ⇨ $N_3 \sim 0$

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Steady State Solution (Home Work Set 6)

$$N_2 - N_1 = \frac{P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21} + (2P + \Gamma_{10})\sigma\phi}$$

Divide top & bottom w/ $P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}$

$$g(\nu) = \frac{g_0(\nu)}{1 + \phi/\phi_{sat}}$$

Saturated Gain

$$g_0(\nu) = \frac{\sigma(\nu)P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}$$

Small Signal Gain

$$\phi_{sat} = \frac{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}{2(P + \Gamma_{10})\sigma(\nu)}, \quad I_{sat} = h\nu\phi_{sat}$$

Saturation Flux

Saturation Intensity

Semi-Classical Laser Theory

Divide top & bottom w/ $P(\Gamma_{10}+\Gamma_{21})+\Gamma_{10}\Gamma_{21}$

$$g(\nu) = \frac{g_0(\nu)}{1+\phi/\phi_{sat}}$$

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$$g_0(\nu) = \frac{\sigma(\nu)P(\Gamma_{10}-\Gamma_{21})N}{P(\Gamma_{10}+\Gamma_{21})+\Gamma_{10}\Gamma_{21}}$$

Small Signal Gain

$$\phi_{sat} = \frac{P(\Gamma_{10}+\Gamma_{21})+\Gamma_{10}\Gamma_{21}}{2(P+\Gamma_{10})\sigma(\nu)}, \quad I_{sat}=h\nu\phi_{sat}$$

Saturation Flux

Saturation Intensity

Gain under Lasing Conditions

Below threshold $g \le g_t \Rightarrow \begin{cases} \phi \le \phi_{sat} \\ g(\nu) \sim g_0(\nu) \end{cases}$

Small Signal Gain

Above threshold: **exp. growth of** ϕ **until the gain saturates, growth slows and stops**

Steady State:

$$g(\nu) = g_t = \kappa/c$$

Saturated Gain = Loss

Important Question:

- **What if many modes see significant gain?**
- **It depends, and can be complicated !**