Vector Model of the 2-Level Atom

We return to the Density Matrix Equations of Motion for a 2-Level atom

$$\dot{\rho}_{11} = -\Gamma_1 \rho_{11} + A_{21}\rho_{22} - \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{22} = -\Gamma_2 \rho_{22} - A_{21}\rho_{22} + \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{12} = (i\Delta - \beta)\rho_{12} + \frac{i\chi^*}{2}(\rho_{22} - \rho_{11}) = \dot{\rho}_{21}^*$$

$$\beta = \frac{1}{T} + \frac{1}{2}(\Gamma_1 + \Gamma_2 + A_{21}), \quad \chi = \vec{\mu}_{21}\cdot\hat{z}E_0/\hbar$$

Note: In our previous iteration we studied the Rate Equation approximation, which is useful when we are looking for steady state solutions

Here our goal is different – we seek to recast the Density Matrix formalism in a way that is better suited to understanding and modeling coherent evolution and transient phenomena. This will also be useful when we study wave and light pulse propagation.

Optical Bloch Equations (OBE's)

Let $\Gamma_1 = \Gamma_2 = 0$ ⇒ $\rho_{11} + \rho_{22} = 1, \; \rho_{12} = \rho_{21}^*$

⇒ 3 independent, real-valued variables

<u>Define</u>	$u = \rho_{21} + \rho_{12}$
Bloch Variables	$v = i(\rho_{21} - \rho_{12})$
	$w = \rho_{22} - \rho_{11}$

Let $\chi = |\chi| e^{-i\varphi}$, substitute in equations for ρ, leaving out relaxation terms $A_{21}, \Gamma_1, \Gamma_2, \beta$

Optical Bloch Equations

$$\dot{u} = -\Delta v - |\chi| \sin\varphi \, w$$

$$\dot{v} = \Delta u + |\chi| \cos\varphi \, w$$

$$\dot{w} = -|\chi|(\cos\varphi \, v - \sin\varphi \, u)$$

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$$\dot{v} = \Delta u + |\chi| \cos\varphi \, w$$

$$\dot{w} = -|\chi| (\cos\varphi \, v - \sin\varphi \, u)$$

Define: $\vec{S} = u\vec{i} + v\vec{j} + w\vec{k}$

$\vec{Q} = -|\chi| \cos\varphi \, \vec{i} - |\chi| \sin\varphi \, \vec{j} + \Delta \vec{k}$

$$\boxed{\dot{\vec{S}} = \underset{\text{Torque}}{\vec{Q}} \times \underset{\text{Bloch Vector}}{\vec{S}}}$$

Equivalent to OBE's

Note: $\frac{d}{dt}(S^2) = 2\vec{S}\cdot\dot{\vec{S}} = 2\vec{S}\cdot(\vec{Q}\times\vec{S}) = 0$

length conserved

From the definition of the Bloch Variables we get

$$\rho = \frac{1}{2}\begin{pmatrix} 1-w & u+iv \\ u-iv & 1+w \end{pmatrix}$$

$$\mathrm{Tr}\,\rho^2 = \frac{1}{2}\left[1 + u^2 + v^2 + w^2\right] = \frac{1}{2}\left[1 + |S|^2\right] \le 1$$

⇨ $\boxed{|S|^2 \le 1}$

Vector Model of the 2-Level Atom

Define:

$$\vec{S} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\vec{Q} = -|X|\cos\varphi\,\vec{i} - |X|\sin\varphi\,\vec{j} + \Delta\vec{k}$$

$$\boxed{\dot{\vec{S}} = \vec{Q} \times \vec{S}}$$

Torque Bloch Vector

Equivalent to OBE's

length conserved

Note: $\frac{d}{dt}(S^2) = 2\vec{S}\cdot\dot{\vec{S}} = 2\vec{S}\cdot(\vec{Q}\times\vec{S}) = 0$

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$$Tr\,\rho^2 = \frac{1}{2}\left[1+u^2+v^2+w^2\right] = \frac{1}{2}\left[1+|S|^2\right] \leq 1$$

$$\Rightarrow \boxed{|S|^2 \leq 1}$$

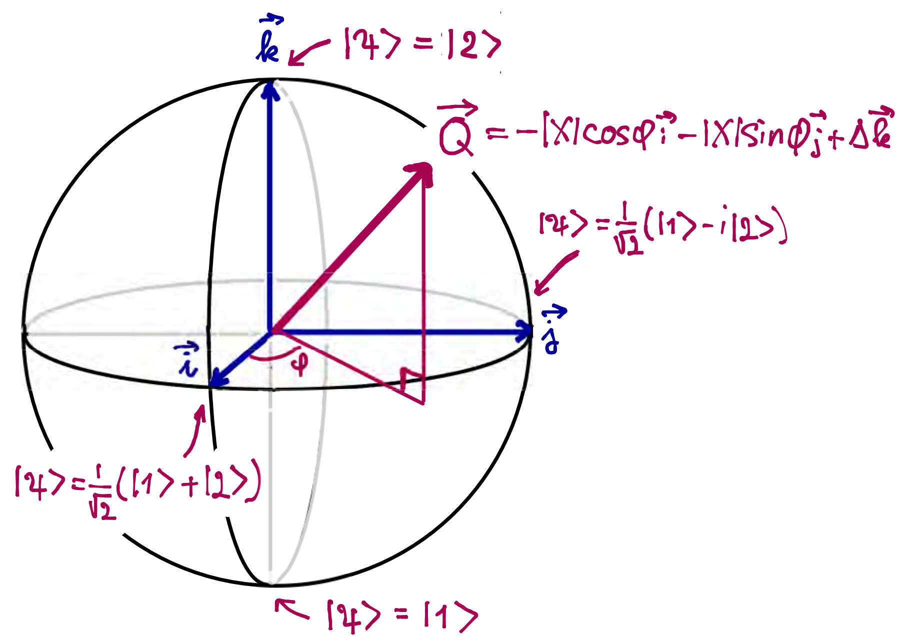
Clearly, $|S^2| = 1 \Rightarrow Tr\,\rho^2 = 1 \Rightarrow$ **pure state**

States w/ $|\vec{S}| < 1 \Rightarrow Tr\,\rho^2 < 1 \Rightarrow$ **mixed state**

$$|\vec{S}| = 0 \Rightarrow Tr\,\rho^2 = 1/2 \Rightarrow \rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

maximally mixed

Note: The above suggests a physical state can be represented by a vector $\vec{S}$, whose tip lies on the surface of (pure) or inside (mixed) a sphere of unit radius, and whose length is conserved under Schrödinger evolution. This is the **Bloch Sphere**.



Vector Model of the 2-Level Atom

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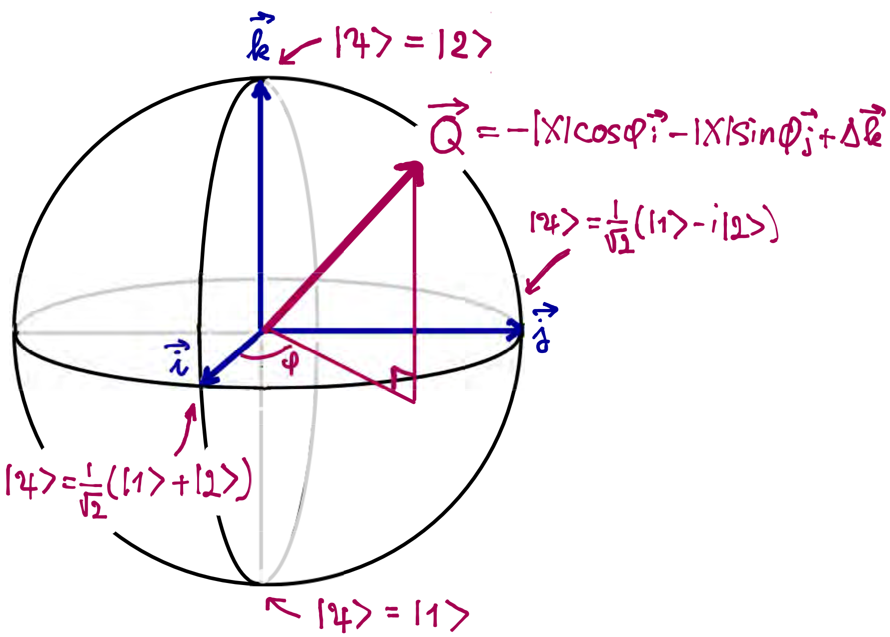

(✱) Do not confuse $\vec{S}$ with the state vector $|\psi\rangle$. $|\psi\rangle$ lives in a complex vector space. Also, do not confuse $\vec{S}$ with a vector in real, physical space. $\vec{S}$ Lives in an abstract, real-valued vector space.

(✱) <u>Only</u> if the 2-level system is a physical spin-1/2 particle does $\vec{S}$ correspond to an angular momentum vector that lives in physical space. In general, $\vec{S}$ is what we call a *pseudo-spin*, not an actual physical spin.

Physical Interpretation of the Bloch Variables

We have $\langle\hat{\vec{\wp}}\rangle = \mathrm{Tr}(\rho\hat{\vec{\wp}}) = \rho_{12}\vec{\wp}_{21} + \rho_{21}\vec{\wp}_{12}$

where

$$\left.\begin{aligned} \rho_{12} &= \tfrac{1}{2}(u+iv)e^{i\omega t} \\ \rho_{21} &= \tfrac{1}{2}(u-iv)e^{-i\omega t} \end{aligned}\right\} \text{fast variables}$$

$$\vec{E} = \mathrm{Re}[\vec{\varepsilon}E_0 e^{-i\omega t}]$$

It follows that

$$\boxed{\begin{aligned} \langle\hat{\vec{\wp}}\rangle &= \tfrac{1}{2}(u+iv)e^{i\omega t}\vec{\wp}_{21} + \tfrac{1}{2}(u-iv)e^{-i\omega t}\vec{\wp}_{12} \\ &= u\,\mathrm{Re}[\vec{\wp}_{12}e^{-i\omega t}] + v\,\mathrm{Im}[\vec{\wp}_{12}e^{-i\omega t}] \end{aligned}}$$

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Thus

u is the component of $\langle\hat{\vec{\eta}}\rangle$ in-phase w/ $\vec{E}$

v is the component of $\langle\hat{\vec{\eta}}\rangle$ in-quadrature w/ $\vec{E}$

Lastly, $w = \rho_{22} - \rho_{11}$ is the population inversion.

Solution of the OBE's

Let $\Delta = 0$ and χ real and positive ⇨ $\begin{cases}\vec{Q} = -|\chi|\vec{i}\\ \varphi = 0\end{cases}$

$$\boxed{\begin{aligned}\vec{S} &= u\vec{i} + v\vec{j} + w\vec{k}\\ \vec{Q} &= -|\chi|\cos\varphi\,\vec{i} - |\chi|\sin\varphi\,\vec{j} + \Delta\vec{k}\end{aligned}}$$

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Choose global phase so u(0)=0

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Solution

Rabi Oscillations

$$\boxed{\begin{aligned} v &= -\sin\theta \\ w &= -\cos\theta \end{aligned} \qquad \theta = \chi t}$$

Vector Model of the 2-Level Atom

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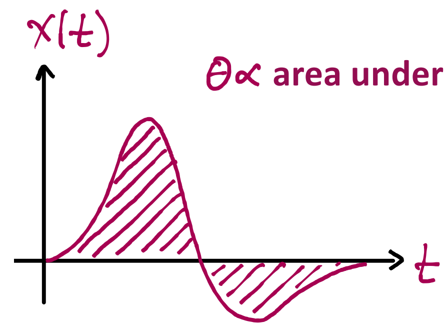
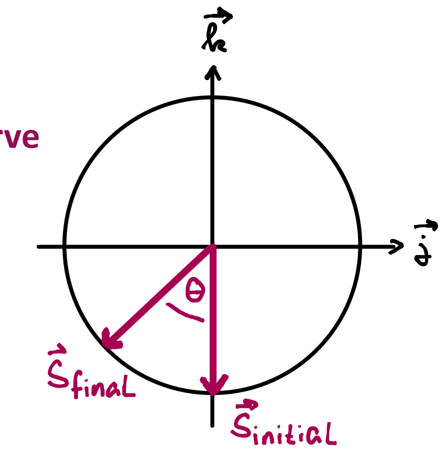
What if $\chi = \chi(t)$ **? We now have solutions**

$$\boxed{\begin{aligned} v &= -\sin\theta \\ w &= -\cos\theta \end{aligned} \qquad \theta = \int_0^t \chi(t')\,dt'}$$

Pulse Area Theorem!

This is a very important result $\Rightarrow$ **We can deal with pulses**





Note: $\chi(t) = \overbrace{\vec{\mu}_{21}\cdot\vec{\varepsilon}}^{\mu}\, E_0/\hbar = \mu\,\mathcal{E}(t)/\hbar$

complex amplitude

Cannot remain real if the complex phase of $\mathcal{E}(t)$ **is changing with time**

Vector Model of the 2-Level Atom

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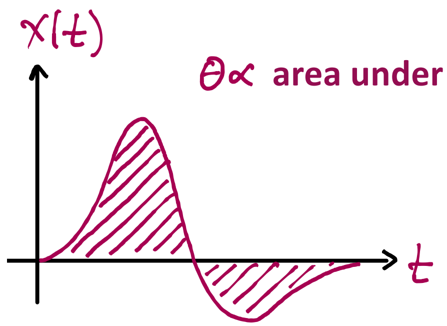
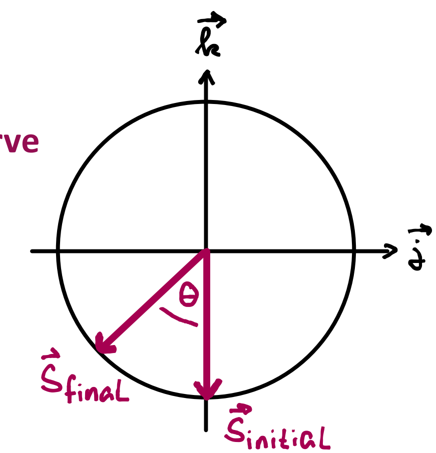
$\theta \propto$ area under curve





Note: $\chi(t) = \overbrace{\vec{p}_{21}\cdot\vec{\varepsilon}}^{\mu}\, E_0/\hbar \equiv \mu\,\mathcal{E}(t)/\hbar$

complex amplitude

Cannot remain real if the complex phase of $\mathcal{E}(t)$ is changing with time

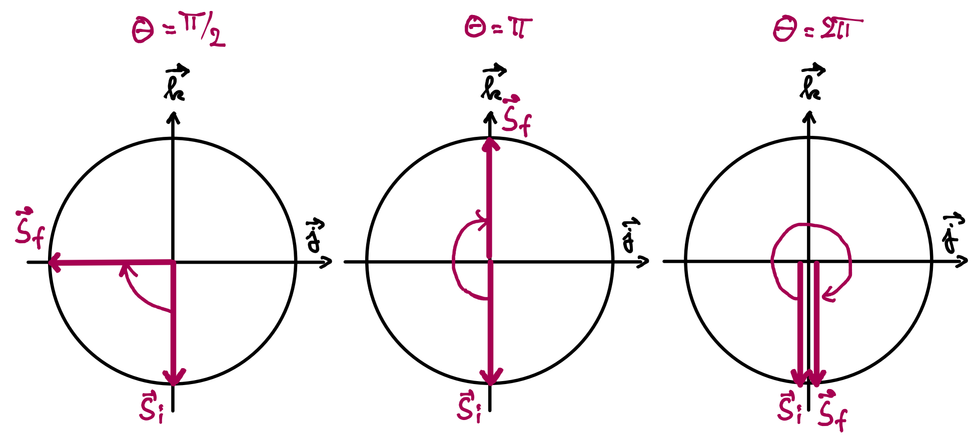
Different phase ➡ different axis of rotation

The Pulse Area Theorem is only valid if the <u>direction</u> of the torque vector $\vec{Q}$ is constant.

Note: The RWA is valid only in the Slowly Varying Envelope Approx. $\left.\right\}$ $\dfrac{d\mathcal{E}}{dt} \ll \omega$

This may not hold for modern Ultrafast Lasers !

Some examples of **θ-pulses**: (quantum gates)



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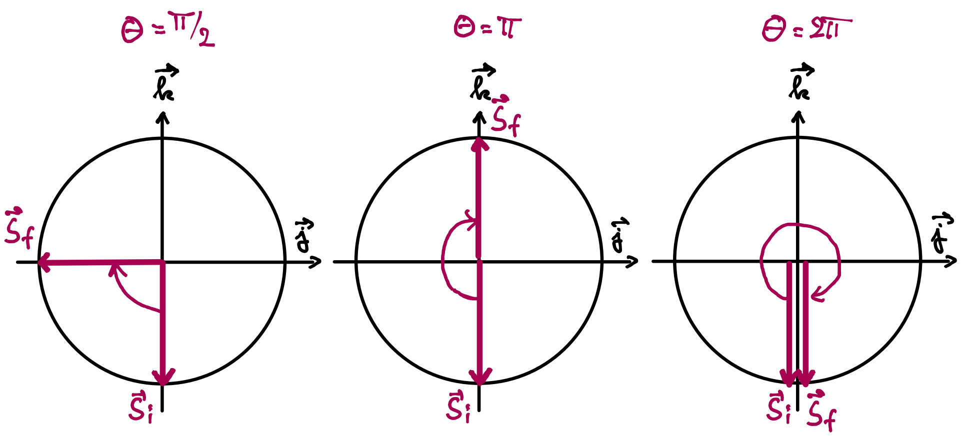
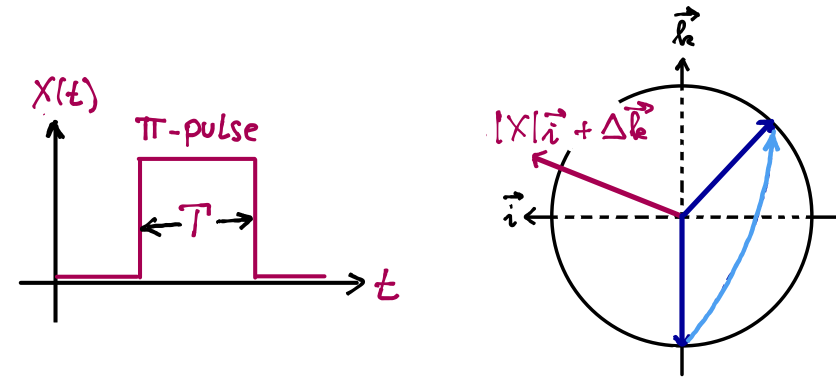


Ramsey Method of Separated Oscillatory Fields

(The Ramsey "trick", 1989 Nobel in Physics)

Single pulse measurement of ω_{21} in Atomic Clock:



Idea is to measure population of $|2\rangle$ as function of $\Delta = \omega_{21} - \omega$, which is maximized for $\omega = \omega_{21}$.

The frequency resolution is $\delta\omega \propto 1/T$, so very long pulses are required. The atom is perturbed by interaction with the light during the entire interrogation and problems can occur due to phase or amplitude noise on the light field. This is not good since the clock is supposed to link to the transition frequency of an unperturbed atom.

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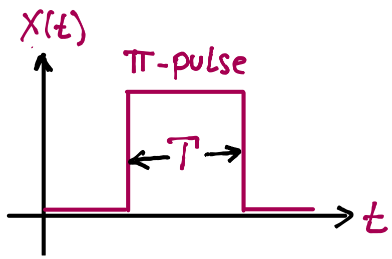

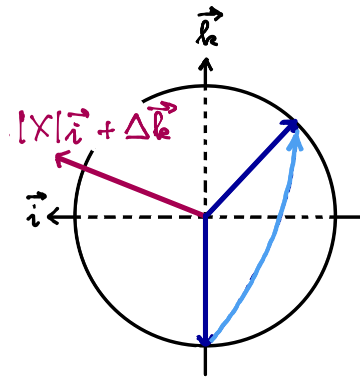

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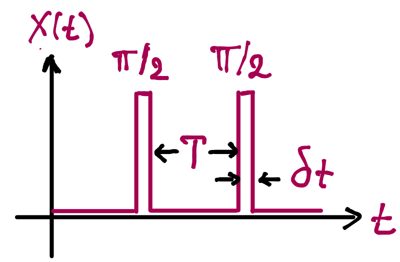
Ramsey's two-pulse strategy:



Sequence of 2 short, intense $\pi/2$ pulses, separated by a long free evolution period, So that $T \gg \delta t$

To be continued…