Problem 1

(a) Following our solution of the Rabi problem in the slides/notes on Atom-Light Interaction in 2-Level Atoms, we get for χ real and $\Delta = 0$ that

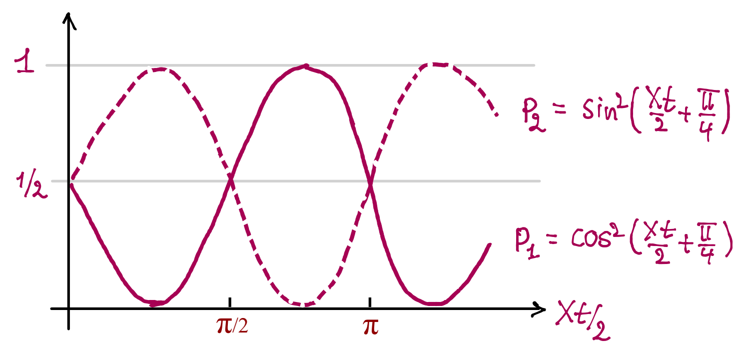
$$c_1(t) = \cos\frac{\chi t}{2}\ , \quad c_2(t) = i\sin\left(\frac{\chi t}{2}\right) \quad \Rightarrow \quad |\varphi\left(\frac{\chi t}{2}\right)\rangle = \cos\frac{\chi t}{2}|1\rangle + i\sin\frac{\chi t}{2}|2\rangle$$

(b) We are looking for a solution with boundary condition

$$|\psi(0)\rangle \;=\; \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle) \;=\; |\varphi\left(\frac{\chi t}{2}+\frac{\pi}{4}\right)\rangle, \quad \text{i. e.,}$$

$$|\psi\left(\frac{\chi t}{2}\right)\rangle \;=\; |\varphi\left(\frac{\chi t}{2}+\frac{\pi}{4}\right)\rangle \quad \Rightarrow \quad \boxed{c_1\left(\frac{\chi t}{2}\right) = \cos\left(\frac{\chi t}{2}+\frac{\pi}{4}\right), \quad c_2\left(\frac{\chi t}{2}\right) = i\sin\left(\frac{\chi t}{2}+\frac{\pi}{4}\right)}$$

It follows that the Rabi oscillations of $P_1(t), P_2(t)$ are as for the problem in the slides/notes, but advanced by 1/4 period.

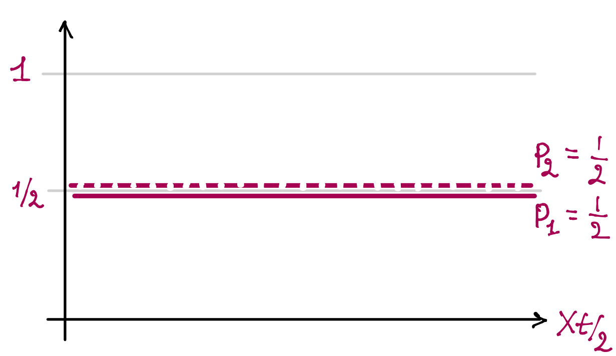


(c) Following the standard recipe, we might compute $P_j = Tr[\rho|\,j\rangle\langle j|]$, $j = 1,2$. However, given $\rho(t=0) = \frac{1}{2}(|1\rangle\langle 1| + |2\rangle\langle 2|)$, we can see directly that we have half the population in $|1\rangle$ and half in $|2\rangle$ at $t = 0$. At later times we have, respectively,

$$\left.\begin{array}{ll}\text{Initially } |1\rangle\langle 1|: & P_1(t) = \frac{1}{2}\cos^2\frac{\chi t}{2},\ P_2(t) = \frac{1}{2}\sin^2\frac{\chi t}{2} \\ \text{Initially } |2\rangle\langle 2|: & P_1(t) = \frac{1}{2}\sin^2\frac{\chi t}{2},\ P_2(t) = \frac{1}{2}\cos^2\frac{\chi t}{2}\end{array}\right\} \Rightarrow P_1(t) \;=\; P_2(t) \;=\; \frac{1}{2}$$

We could also argue that $\rho(t=0) = \frac{1}{2}\mathrm{I}$ (I is the identity), and therefore $i\hbar\dot{\rho} = [H,\rho] = 0$ and ρ is constant in time. That means $P_1(t), P_2(t)$ are constant too.

The plot is pretty boring:



Problem 2

(a) From the notes and slides about the Rate Equation formalism, we have

$$\Phi_{sat} = \frac{\beta}{\sigma(0)} = \frac{2\pi\beta}{3\lambda^2} = \frac{2\pi\times 10^7/\text{s}}{3(2\times 10^{-6}\text{m})^2} \doteq \underline{\underline{5.24\times 10^{18}/\text{m}^2\text{s}}}$$

$$I_{sat} = \hbar\omega\Phi_{sat} = 1.05\times 10^{-34}\text{J s}\times 10^{15}/\text{s}\times 5.24\times 10^{18}/\text{m}^2\text{s} \doteq \underline{0.55\ \text{W/m}^2}$$

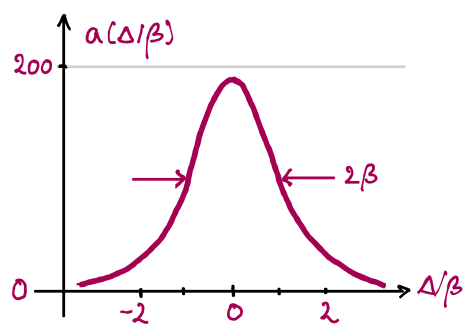
(b) Again we have, directly from the notes and slides about the Rate Equation formalism, the folowing expression for the absorption coefficient when $I << I_{sat}$:

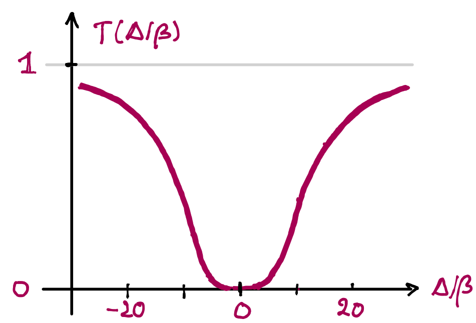
$$\underline{a(\Delta) = N\sigma(\Delta)} = N\sigma(0)\frac{\beta^2}{\Delta^2+\beta^2} = N\frac{3\lambda^2}{2\pi}\frac{1}{(\Delta/\beta)^2+1} \doteq \underline{191/\text{m}\times\frac{1}{(\Delta/\beta)^2+1}}$$

$$\underline{T(\Delta) = e^{-a(\Delta)L}}$$

As instructed, we calculate $T(\Delta)$ at sample points in the range $\Delta/\beta \in [0,20]$ and then draw the sketches:

Δ/β	T
0	3.3×10^{-42}
2	5.1×10^{-9}
5	0.025
6	0.076
8	0.23
10	0.39
15	0.66
20	0.79





(c) The transmission versus detuning dip looks like it has a FWHM of $\sim 25\beta$ and the curve is almost flat over an interval of $\sim \pm 5\beta$ around resonance. That makes it difficult to locate the center of the transmission line. The problem is that the transmission $T(\Delta) = e^{-a(\Delta)L}$ is not a linear function of $a(\Delta)L$. One option is to lower the number density N or shorten the optical path length L until the medium is no longer optically thick on resonance. Once $e^{-a(\Delta)L} \approx 1 - a(\Delta)L$ the shape of the transmission dip will look more like an inverted Lorentzian with a FWHM $\sim 2\beta$. Note that if we try to render the medium more transparent by turning up the intensity so $I >> I_{sat}$, we end up with a power broadened linewidth $\sim \beta\sqrt{1 + I/I_{sat}}$ which is also detrimental.