

OPTI 544 Solution Set 5, Spring 2021

Problem 1

- (a) Because χ is real and positive ($\varphi = 0$) we have a torque vector $\vec{Q} = -\chi\vec{i}$ and the Bloch vector \vec{S} precesses in the $\vec{j} - \vec{k}$ plane. At $t = \pi/\chi_0$ the precession angles for the two atoms are $3\pi/4$ and $5\pi/4$, respectively.

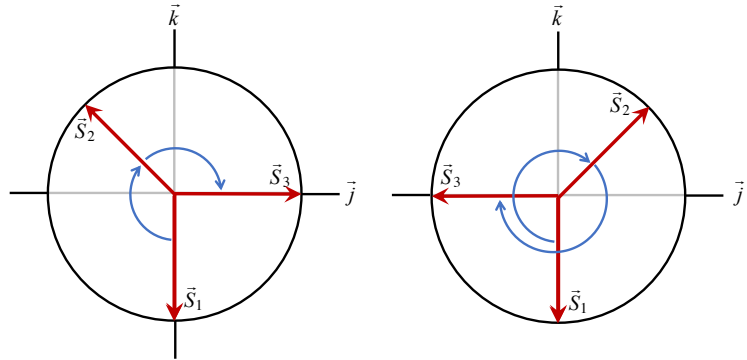
Bloch Vectors

\vec{S}_1 at $t = 0$

\vec{S}_2 at $t = \pi/\chi_0$

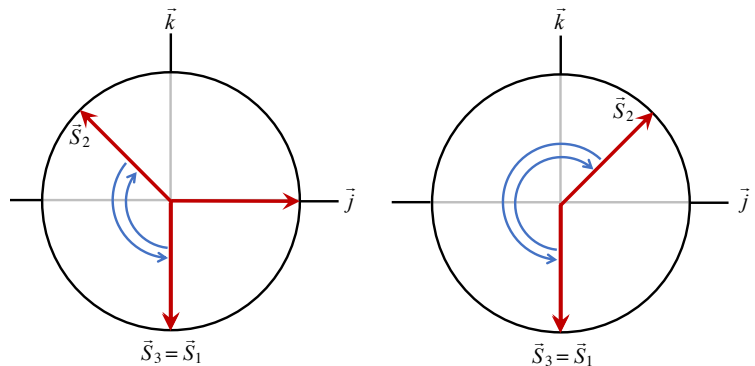
\vec{S}_3 at $t = 2\pi/\chi_0$

Showing $\vec{j} - \vec{k}$ plane,
 \vec{i} points into the page



- (b) Changing phase by 180° changes $\chi \rightarrow -\chi$ and thus the direction of precession, but the rate of precession is unchanged. The second pulse thus undoes the precession during the first pulse irrespective of the value of χ , and thus returns the Bloch vector to its initial state.

Bloch Vectors



Problem 2

Before we start, we adapt the shorthand notation $\frac{\partial}{\partial z} \mathcal{E} = \mathcal{E}'$, $\frac{\partial}{\partial t} \mathcal{E} = \dot{\mathcal{E}}$ and so on. (\mathcal{E} is the best I can do for \mathcal{E} in my equation editor)

(a) We start from the wave equation in a polarizable dielectric medium,

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{E}(z, t) = \frac{1}{\epsilon_0 c^2} \frac{\partial^2}{\partial t^2} \mathbf{P}(z, t)$$

We plug in $\mathbf{E}(z, t) = \bar{\epsilon} \mathcal{E}(z, t) e^{-i(\omega t - kz)}$ and $\mathbf{P}(z, t) = \bar{\epsilon} 2N\mu^* \rho_{21}(z, t) e^{-i(\omega t - kz)}$, take the scalar product with $\bar{\epsilon}$ on both sides to remove the vector character, and expand out the derivatives. For the LHS we get

$$\begin{aligned} \mathcal{E}'' e^{-i(\omega t - kz)} + 2ik\mathcal{E}' e^{-i(\omega t - kz)} - k^2 \mathcal{E} e^{-i(\omega t - kz)} - \frac{1}{c^2} \ddot{\mathcal{E}} e^{-i(\omega t - kz)} + \frac{i2\omega}{c^2} \dot{\mathcal{E}} e^{-i(\omega t - kz)} + \frac{\omega^2}{c^2} \mathcal{E} e^{-i(\omega t - kz)} \\ = 2ik \left(\mathcal{E}' + \frac{1}{c^2} \dot{\mathcal{E}} \right) e^{-i(\omega t - kz)} \end{aligned}$$

where, according to the SVEA, we have dropped the \mathcal{E}'' and $\ddot{\mathcal{E}}$ terms, and used $k^2 - \omega^2/c^2 = 0$ to simplify. Note that the latter does not preclude a complex index of refraction, as the effect of absorption and dispersion will be incorporated into the complex envelope $\mathcal{E}(z, t)$.

On the RHS of the wave equation we have

$$\frac{1}{\epsilon_0 c^2} \frac{\partial^2}{\partial t^2} \mathbf{P}(z, t) = \frac{2N\mu^*}{\epsilon_0 c^2} (\ddot{\rho}_{21} - i2\omega \dot{\rho}_{21}) e^{-i(\omega t - kz)} = -\frac{2k^2 N\mu^*}{\epsilon_0} \rho_{21}(z, t) e^{-i(\omega t - kz)}.$$

where we have dropped the $\ddot{\rho}_{21}$ and $\dot{\rho}_{21}$. Putting everything together, dividing out the plane wave component, and rearranging, we finally get

$$\boxed{\left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right) \mathcal{E}(z, t) = \frac{ik}{\epsilon_0} N\mu^* \rho_{21}(z, t)}$$

(b) In steady state we substitute $\rho_{21}(\infty) = -i \frac{\chi}{2} \frac{\beta - i\Delta}{\Delta^2 + \beta^2} (\rho_{22} - \rho_{11})$ (from Homework Set 3)

We also set $\dot{\mathcal{E}} = 0$ and use $\chi = \bar{p}_{21} \cdot \bar{\epsilon} \mathcal{E} / \hbar = \mu \mathcal{E} / \hbar$. This gives us

$$\frac{\partial}{\partial z} \mathcal{E} = \frac{kN|\mu|^2}{2\varepsilon_0} \frac{\beta - i\Delta}{\Delta^2 + \beta^2} (\rho_{22} - \rho_{11}) \mathcal{E} = \frac{1}{2} (a - i\delta) (\rho_{22} - \rho_{11}) \mathcal{E}$$

$$\text{where } a = \frac{kN|\mu|^2}{\varepsilon_0 \hbar} \frac{\beta}{\Delta^2 + \beta^2}, \quad \delta = \frac{kN|\mu|^2}{\varepsilon_0 \hbar} \frac{\Delta}{\Delta^2 + \beta^2}$$

These are the results from Class, from my handwritten notes, and from Milloni and Eberly.

Problem 3

(a) We have

$$\begin{aligned}\langle \hat{p} \rangle &= \text{Tr}[\hat{p} \hat{\rho}] = \text{Tr} \left[\begin{pmatrix} 0 & \bar{p}_{12} \\ \bar{p}_{21} & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} e^{i\omega t} \\ \rho_{21} e^{-i\omega t} & \rho_{22} \end{pmatrix} \right] = \text{Tr} \left[\begin{pmatrix} \bar{p}_{12} \rho_{21} e^{-i\omega t} & \bar{p}_{12} \rho_{22} \\ \bar{p}_{21} \rho_{11} & \bar{p}_{21} \rho_{12} e^{i\omega t} \end{pmatrix} \right] \\ &= \bar{p}_{12} \rho_{21} e^{-i\omega t} + \bar{p}_{21} \rho_{12} e^{i\omega t} = \text{Re}[2\bar{p}_{12} \rho_{21} e^{-i\omega t}]\end{aligned}$$

In steady state we have $\rho_{21} = \frac{\chi}{2} \frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}}$ (from Problem 2-b above)

We plug this in above and get $\langle \hat{p} \rangle = \text{Re} \left[\bar{p}_{12} \frac{\beta + i\Delta}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}} \chi e^{-i\omega t} \right]$.

Now, if $A_{21} = 2\beta$ we can use $\frac{|\chi|^2}{A_{21}} = \frac{I}{I_{\text{Sat}}}$ to rewrite the above as

$$\langle \hat{p} \rangle = \text{Re} \left[\bar{p}_{12} \frac{\Delta + i\beta}{\Delta^2 + \beta^2 (1 + I/I_{\text{Sat}})} \chi e^{-i\omega t} \right]$$

(b) Noting that $\chi = \bar{p}_{21} \cdot \bar{\mathcal{E}} E / \hbar = \mu E / \hbar$, where $\mu = \bar{p}_{21} \cdot \bar{\mathcal{E}}$, we have

$$\vec{p}'(t) = \bar{p}_{12} \frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}} \chi e^{-i\omega t} = \frac{\bar{p}_{12} \mu}{\hbar} \frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}} E e^{-i\omega t}$$

The part of $\vec{p}'(t)$ parallel to \mathbf{E} is (remember $\bar{\mathcal{E}}$ is complex)

$$\bar{p}(t) = (\vec{p}'(t) \cdot \bar{\mathcal{E}}^*) \bar{\mathcal{E}} = \frac{|\mu|^2}{\hbar} \frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}} \bar{\mathcal{E}} E e^{-i\omega t} \equiv \alpha(\omega) \mathbf{E}(t)$$

We thus obtain

$$\alpha(\omega) = \frac{|\mu|^2}{\hbar} \frac{\Delta + i\beta}{\Delta^2 + \beta^2 + |\chi|^2 \beta / A_{21}} = \frac{|\mu|^2}{\hbar} \frac{\Delta + i\beta}{\Delta^2 + \beta^2 (1 + I/I_{\text{Sat}})}$$

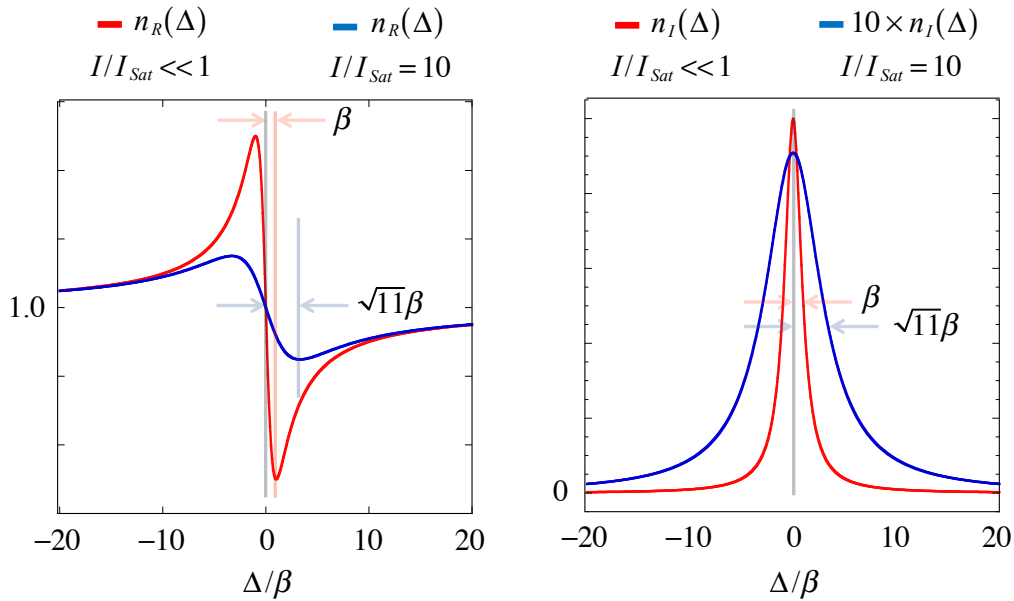
where the second step is true when $A_{21} = 2\beta$. We will assume this is the case in part (c).

(c) For $n(\omega) \approx 1$ we have

$$n_R(\omega) = 1 + \frac{N}{2\epsilon_0} \operatorname{Re}[\alpha(\omega)] = 1 + \frac{N|\mu|^2}{2\epsilon_0\hbar} \frac{\Delta}{\Delta^2 + \beta^2(1 + I/I_{Sat})}$$

$$n_I(\omega) = \frac{N}{2\epsilon_0} \operatorname{Im}[\alpha(\omega)] = \frac{N|\mu|^2}{2\epsilon_0\hbar} \frac{\beta}{\Delta^2 + \beta^2(1 + I/I_{Sat})}$$

Sketch for $I/I_{Sat} \ll 1$ and for $I/I_{Sat} = 10$:



(d) For $I/I_{Sat} \gg 1$ both the dispersion and absorption features are broadened. In the sketch we have set $I/I_{Sat} = 10$, which makes the power broadened linewidth

$$\beta' = \beta\sqrt{1 + I/I_{Sat}} = \sqrt{11}\beta \sim 3.3\beta$$

At the same time, the peak dispersion is reduced by a factor

$$1/\sqrt{1 + I/I_{Sat}} = 1/\sqrt{11} = 0.30$$

And the peak absorption is reduced by a factor

$$1/(1 + I/I_{Sat}) = 1/11 = 0.091.$$