

OPTI 544 Solution Set 3, Spring 2021

Problem 1

- (a) The equations of motion for the probability amplitudes in the RWA, using the “slow” variables and setting $\Delta = \delta = 0$, are

$$\dot{b}_1 = -i\frac{\chi_1}{2}b_2, \quad \dot{b}_2 = -i\left(\frac{\chi_1}{2}b_1 + \frac{\chi_2}{2}b_3\right), \quad \dot{b}_3 = -i\frac{\chi_2}{2}b_2.$$

- (b) If $b_2(0) = 0$ then $b_2(t) = 0$ if and only if $\dot{b}_2 = 0$ at all times. That requires

$$\left(\frac{\chi_1}{2}b_1 + \frac{\chi_2}{2}b_3\right) = 0 \Rightarrow b_3 = -\frac{\chi_1}{\chi_2}b_1.$$

Now $|b_1|^2 + |b_3|^2 = 1 \Rightarrow |b_1|^2 + \frac{\chi_1^2}{\chi_2^2}|b_1|^2 = 1 \Rightarrow |b_1|^2 = \frac{\chi_2^2}{\chi_1^2 + \chi_2^2}.$

Consistent with the above, we choose

$$b_1 = \frac{\chi_2}{\sqrt{\chi_1^2 + \chi_2^2}}, \quad b_3 = \frac{-\chi_1}{\sqrt{\chi_1^2 + \chi_2^2}}$$

Note: If $b_2(t) = 0$ then $\dot{b}_1 = \dot{b}_3 \Rightarrow b_1, b_3$ are constant

- (c) If the driving field does not lead to a non-zero probability amplitude in the excited state $|2\rangle$, then there can be no induced dipole moment. This is because $|1\rangle$ and $|3\rangle$ of necessity must be of the same parity to allow for Raman coupling in the first place. That means there can be no absorption or emission of light, and as a result the wave propagates without loss of intensity.

Note: The state found in (b) above is referred to as a “dark state”.

Problem 2

(a) Taking the outer product of the state vectors we find

$$\rho_1 = \frac{1}{3} \begin{pmatrix} 1 \\ i\sqrt{2} \\ 0 \end{pmatrix} (1, -i\sqrt{2}, 0) = \frac{1}{3} \begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{Check: } \text{Tr}[\rho_1] = 1$$

$$\rho_2 = \frac{1}{5} \begin{pmatrix} 1+i \\ 1 \\ -i\sqrt{2} \end{pmatrix} (1-i, i\sqrt{2}) = \frac{1}{5} \begin{pmatrix} 2 & 1+i & -(1-i) \\ 1-i & 1 & i\sqrt{2} \\ -(1+i) & -i\sqrt{2} & 2 \end{pmatrix}. \quad \text{Check: } \text{Tr}[\rho_2] = 1$$

$$\begin{aligned} \text{Then } \rho &= \frac{1}{2}(\rho_1 + \rho_2) = \frac{1}{2} \times \frac{1}{15} \left[\begin{pmatrix} 1 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1+i & -(1-i) \\ 1-i & 1 & i\sqrt{2} \\ -(1+i) & -i\sqrt{2} & 2 \end{pmatrix} \right] \\ &= \frac{1}{30} \begin{pmatrix} 11 & 3+i(3-5\sqrt{2}) & -(1-i)3\sqrt{2} \\ 3-i(3-5\sqrt{2}) & 13 & i3\sqrt{2} \\ -(1+i)3\sqrt{2} & -i3\sqrt{2} & 6 \end{pmatrix}. \quad \text{Check: } \text{Tr}[\rho] = 1 \end{aligned}$$

(b) To check for purity we can compute ρ^2 and then check if $\rho^2 \neq \rho$ or $\text{Tr}[\rho^2] < 1$, either of which would tell us that the state is mixed. For example, in this case it is straightforward though somewhat tedious to show that $\text{Tr}[\rho^2] = (19 - 2\sqrt{2})/30 = 0.539 < 1$, which tells us the state is mixed.

There is an easier way if we know the ensemble decomposition, $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. Namely, that ρ is mixed if one or more of the $|\psi_k\rangle$ are linearly independent of the other. In our case we can confirm this by inspection, since $|\psi_1\rangle$ is confined to the $\{|1\rangle, |2\rangle\}$ subspace while $|\psi_2\rangle$ has a component along $|3\rangle$.

Problem 3

(a) The Hamiltonian for our 2-level system has the form

$$H = \hbar \begin{pmatrix} 0 & -\frac{1}{2}(\chi_{12} e^{-i\omega t} + \chi_{21}^* e^{i\omega t}) \\ -\frac{1}{2}(\chi_{21} e^{-i\omega t} + \chi_{12}^* e^{i\omega t}) & \omega_{21} \end{pmatrix}$$

Starting from $i\hbar\dot{\rho} = [H, \rho]$ we get $i\hbar\dot{\rho}_{ij} = \sum_k (H_{ik}\rho_{kj} - \rho_{ik}H_{kj})$, $i, j, k \in \{1, 2\}$.

Applying this to the elements of the 2-level density matrix, we get

$$\begin{aligned} \dot{\rho}_{11} &= \frac{1}{i\hbar} (H_{11}\rho_{11} - \rho_{11}H_{11} + H_{12}\rho_{21} - \rho_{12}H_{21}) \\ &= \frac{i}{2}(\chi_{12}e^{-i\omega t} + \chi_{21}^*e^{i\omega t})\rho_{21} - \frac{i}{2}(\chi_{21}e^{-i\omega t} + \chi_{12}^*e^{i\omega t})\rho_{12} \end{aligned}$$

and
$$\begin{aligned} i\hbar\dot{\rho}_{12} &= H_{11}\rho_{12} - \rho_{11}H_{12} + H_{12}\rho_{22} - \rho_{12}H_{22} \\ &= i\omega_{21}\rho_{12} + \frac{i}{2}(\chi_{12}e^{-i\omega t} + \chi_{21}^*e^{i\omega t})(\rho_{22} - \rho_{11}) \end{aligned}$$

Now let $\rho_{12} = \tilde{\rho}_{12}e^{i\omega t}$, $\rho_{21} = \tilde{\rho}_{21}e^{-i\omega t}$ and substitute in the above equations. This gives us

$$\begin{aligned} \dot{\rho}_{11} &= \frac{i}{2}(\chi_{12}e^{-i\omega t} + \chi_{21}^*e^{i\omega t})\rho_{21} - \frac{i}{2}(\chi_{21}e^{-i\omega t} + \chi_{12}^*e^{i\omega t})\rho_{12} \\ \Rightarrow \dot{\rho}_{11} &= \frac{i}{2}(\chi_{12}e^{-2i\omega t} + \chi_{21}^*)\tilde{\rho}_{21} - \frac{i}{2}(\chi_{21} + \chi_{12}^*e^{2i\omega t})\tilde{\rho}_{12} \\ \dot{\rho}_{12} &= \frac{d}{dt}(\tilde{\rho}_{12}e^{i\omega t}) = \dot{\tilde{\rho}}_{12}e^{i\omega t} + i\omega\tilde{\rho}_{12}e^{i\omega t} = i\omega_{21}\tilde{\rho}_{12}e^{i\omega t} + \frac{i}{2}(\chi_{12}e^{-i\omega t} + \chi_{21}^*e^{i\omega t})(\rho_{22} - \rho_{11}) \\ \Rightarrow \dot{\tilde{\rho}}_{12} &= i(\omega_{21} - \omega)\tilde{\rho}_{12} + \frac{i}{2}(\chi_{12}e^{-i2\omega t} + \chi_{21}^*)(\rho_{22} - \rho_{11}) \end{aligned}$$

We set $\Delta = (\omega_{21} - \omega)$, $\chi_{12} = \chi$, $\chi_{21}^* = \chi^*$, drop the terms $\propto e^{\pm i2\omega t}$, and use $\dot{\rho}_{22} = -\dot{\rho}_{11}$, $\dot{\rho}_{12} = \dot{\tilde{\rho}}_{21}^*$. This gives us the desired result

$\begin{aligned} \dot{\rho}_{11} &= -\frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21}), & \dot{\rho}_{22} &= \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21}) \\ \dot{\rho}_{12} &= i\Delta\rho_{12} + \frac{i\chi^*}{2}(\rho_{22} - \rho_{11}), & \dot{\rho}_{21} &= -i\Delta\rho_{21} - \frac{i\chi^2}{2}(\rho_{22} - \rho_{11}) \end{aligned}$
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- (b) The Density Matrix has 2 real-valued populations and 2 complex-valued coherences, which suggests a total of 6 real-valued variables that must be known in order to specify ρ . However, the constraints $\rho_{22} = 1 - \rho_{11}$ and $\rho_{12} = \rho_{21}^*$ allow us to express 3 of the 6 variables in terms of the other 3. This leaves us with a total of 3 real-valued variables necessary to specify an arbitrary Density Matrix, whether it is pure or mixed.
- (c) Major approximations implicit in the above result:
- (i) The Electric Dipole Approximation (inherent in the form of H)
 - (ii) The 2-Level Approximation
 - (iii) The Rotating Wave Approximation

Problem 4

In steady state the Density Matrix Equations reduce to

- (i) $\dot{\rho}_{11} = A_{21}\rho_{22} - \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21}) = 0$
- (ii) $\dot{\rho}_{22} = -A_{21}\rho_{22} + \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21}) = 0$
- (iii) $\dot{\rho}_{12} = -(\beta - i\Delta)\rho_{12} + i\frac{\chi^*}{2}(\rho_{22} - \rho_{11}) = \dot{\rho}_{21}^* = 0$

We start by solving for the coherences in Equation (iii)

$$\rho_{12} = \frac{i\chi^*}{2}(\rho_{22} - \rho_{11}) = i\chi^* \frac{\beta + i\Delta}{\Delta^2 + \beta^2}(\rho_{22} - \rho_{11}), \quad \rho_{21} = -i\frac{\chi}{2} \frac{\beta - i\Delta}{\Delta^2 + \beta^2}(\rho_{22} - \rho_{11})$$

From this we get $\chi\rho_{12} - \chi^*\rho_{21} = \frac{i|\chi|^2\beta}{\Delta^2 + \beta^2}(\rho_{22} - \rho_{11})$

Substituting in Equation (ii), using $\rho_{22} - \rho_{11} = 2\rho_{22} - 1$, and solving for ρ_{22} , we get

$$\begin{aligned} \rho_{22} &= \frac{i}{2A_{21}} \frac{i|\chi|^2\beta}{\Delta^2 + \beta^2} (2\rho_{22} - 1) \Rightarrow \left(1 + \frac{|\chi|^2\beta / A_{21}}{\Delta^2 + \beta^2} \right) \rho_{22} = \frac{1}{2} \frac{|\chi|^2\beta / A_{21}}{\Delta^2 + \beta^2} \\ \Rightarrow \rho_{22} &= \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} \end{aligned}$$

Next,

$$\rho_{11} = 1 - \rho_{22} = 1 - \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} = \frac{\Delta^2 + \beta^2 + |\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}}$$

Sanity check: $\rho_{11} + \rho_{22} = 1$. With that we have the steady state solutions

$$\begin{aligned} \rho_{11}(\infty) &= \frac{\Delta^2 + \beta^2 + |\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} & \rho_{22}(\infty) &= \frac{|\chi|^2\beta / 2A_{21}}{\Delta^2 + \beta^2 + |\chi|^2\beta / A_{21}} \\ \rho_{12}(\infty) &= i\frac{\chi^*}{2} \frac{\beta + i\Delta}{\Delta^2 + \beta^2}(\rho_{22} - \rho_{11}) & \rho_{21}(\infty) &= -i\frac{\chi}{2} \frac{\beta - i\Delta}{\Delta^2 + \beta^2}(\rho_{22} - \rho_{11}) \end{aligned}$$