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Optical Physics

Light –Matter Interactions

We have a hierarchy of descriptions at increasing sophistication

- Classical	Classical light, classical matter
- Semiclassical	Classical light, quantum matter
- Quantum	Quantum light, quantum matter

When choosing a description, there are several possible philosophies

Purist:	Always use most complete description possible
Minimalist:	Only use quantum mechanics when necessary (pedantic)
Pragmatic:	Use quantum or classical description, based on whatever is simplest and works.

In this course we study classical, semiclassical and fully quantum descriptions in turn, with the pragmatic attitude that one will most often use the simplest theory that works for the problem at hand.

Classical Theory of Light –Matter Interaction

Light affects the particles that make up the medium, and the medium affects the light. Our goal:

Self-consistent, fully classical description

$$\begin{tikzcd} \text{Electromagnetic field} \arrow[r] & \text{Atom/molecule/solid} \arrow[l, bend left] \end{tikzcd}$$

Motivation:

We will

- Develop concepts ($\alpha(\omega), n, \chi$)
- Develop intuition (useful later for quantum description)
- A classical description is often adequate, and frequently accurate
- A quantum theory has classical limits, identifying regime of validity
- The classical description is a useful jump-off point for non-linear optics

Classical Linear Optics:	Milonni & Eberly chapters 2 & 3
Classical Nonlinear Optics:	Milonni & Eberly chapters 17

The Electromagnetic Field Basic equations in SI units

Maxwell's Equations: (no free charges, no currents $\rightarrow$ dielectrica)

(i)	$\nabla \cdot \mathbf{D} = \rho \equiv 0$	$\mathbf{D}$: Dielectric displacement
(ii)	$\nabla \cdot \mathbf{B} \equiv 0$	$\mathbf{B}$: Magnetic induction
(iii)	$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$	$\mathbf{E}$: Electric field
(iv)	$\nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t + \mathbf{J} = \partial \mathbf{D} / \partial t$	$\mathbf{H}$: Magnetic field

Material Response

$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$ $\longleftarrow$ non-magnetic $\Rightarrow$ $\mathbf{M} = 0$

$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ $\longleftarrow$ info about response in macroscopic dipole moment density, polarization density

We need equations that describe

- the behavior of $\mathbf{E}$ for a given $\mathbf{P}$
- the medium response $\mathbf{P}$ for a given $\mathbf{E}$

Derivation of the wave equation

Take the curl of (iii), then use (iv)

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}$$

Next, use the identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ to obtain

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}$$

Finally, let $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ and use $\varepsilon_0 \mu_0 = 1/c^2$ to obtain

$$\boxed{-\nabla(\nabla \cdot \mathbf{E}) + \nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

This is the wave equation, still exact in this form.

Transverse fields

The definition of a transverse field is that $\nabla \cdot \mathbf{E} \equiv 0$.

This is true for example for a plane wave, $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}}$, $\mathbf{E}(t) \perp \mathbf{k}$, where $\mathrm{Re}[\mathbf{E}(\mathbf{r},t)]$ is the *physical field*. The wave equation can now be simplified to

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

Here, the polarization density $\mathbf{P}$ is a source term for the field that arise due to the response of the medium.

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Note: This version of the wave equation can be a poor approximation in non-isotropic media

Isotropic media

In the absence of a preferred direction the induced polarization **P** must be parallel to the driving field **E**. In the regime of linear response and the otherwise most general case we have

$$\mathbf{D}(t) = \varepsilon_0 \mathbf{E}(t) + \mathbf{P}(t) = \varepsilon_0 \mathbf{E}(t) + \varepsilon_0 \int_{-\infty}^{t} dt' R(t-t') \mathbf{E}(t')$$

where $R(t-t')$ is the response function that describes the memory that the medium has of the history of the field. We have $R(\tau) = 0$ for $\tau < 0$, and R must be a scalar function if the medium is isotropic.

Now take the divergence on both sides and use Maxwell eq. (i)

$$\nabla \cdot \mathbf{D}(t) = \varepsilon_0 \nabla \cdot \mathbf{E}(t) + \varepsilon_0 \int_{-\infty}^{t} dt' R(t-t') \nabla \cdot \mathbf{E}(t') \equiv 0 \Rightarrow$$
$$\nabla \cdot \mathbf{E}(t) = -\int_{-\infty}^{t} dt' R(t-t') \nabla \cdot \mathbf{E}(t') \quad \text{for all } t$$

It follows that $\nabla \cdot \mathbf{E}(t) \equiv 0$ for all t, or $R(\tau) = -2\delta(\tau) \Rightarrow \mathbf{D}(t) = 0$.

Note

- If $R(\tau) \propto \delta(\tau)$ (instantaneous response) then $\varepsilon_0 \int_{-\infty}^{t} dt' R(t-t') \mathbf{E}(t') = \varepsilon_0 \chi \mathbf{E}(t)$, where χ is the electric susceptibility.
- $R(\tau) = -2\delta(\tau)$ is an example of negative susceptibility, $\chi < 0$, which does not occur except in certain engineered metamaterials.

We conclude:

Electric fields are always transverse in linear, isotropic dielectric media.

Note: This does not apply to crystal optics, as many crystalline materials are NOT isotropic. Also not to inhomogeneous media, in particular waveguides.

Wave Equation in Free Space

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0}$$

We adopt a monochromatic trial solution $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})\, e^{-i\omega t}$

$$\nabla^2 \mathbf{E}_0(\mathbf{r}) e^{-i\omega t} + \frac{\omega^2}{c^2}\mathbf{E}_0(\mathbf{r}) e^{-i\omega t} = 0$$

This gives us an equation for the spatial component alone

$$\boxed{\nabla^2 \mathbf{E}_0(\mathbf{r}) + k^2 \mathbf{E}_0(\mathbf{r}) = 0,\ \ k \equiv \omega/c}$$

This equation has plane wave solutions

$$\boxed{\mathbf{E}_0(\mathbf{r}) = \hat{\varepsilon} E_0 e^{i\,\mathbf{k}\cdot\mathbf{r}},\ \ |\mathbf{k}| \equiv \omega/c}$$

Note that an understanding of plane wave propagation is broadly applicable, since any arbitrary field $\mathbf{E}_0(\mathbf{r})$ has a plane wave decomposition.

Wave Equation in Optical Cavities

Optical cavities are an important special case, for which we need to solve the wave equation subject to boundary conditions. See Milonni & Eberly p. 23-27 for the case of a rectangular cavity, Appendix 11.A for Fabryt Perot Etalons, and Chapter 14 for spherical mirror laser resonators.

Wave Equation in ω,k space (Fourier domain)

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}$$

$$\mathbf{E}(\mathbf{r},t) \equiv \int_{\Re} d\omega\, e^{i\omega t}\int_{\Re^3} d^3k e^{i\,\mathbf{k}\cdot\mathbf{r}}\ \mathbf{E}(\mathbf{k},\omega)$$

$$\mathbf{P}(\mathbf{r},t) \equiv \int_{\Re} d\omega\, e^{i\omega t}\int_{\Re^3} d^3k e^{i\,\mathbf{k}\cdot\mathbf{r}}\ \mathbf{P}(\mathbf{k},\omega)$$

Note: $\mathbf{E}(\mathbf{k},\omega)$ and $\mathbf{P}(\mathbf{k},\omega)$ are the temporal and spatial Fourier Transforms of $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{P}(\mathbf{r},t)$.

We substitute into the wave equation, and use

$$\nabla^2 e^{i\,\mathbf{k}\cdot\mathbf{r}}\ \mathbf{E}(\mathbf{k},\omega) = -k^2 e^{i\,\mathbf{k}\cdot\mathbf{r}}\ \mathbf{E}(\mathbf{k},\omega), \qquad \frac{\partial^2}{\partial t^2} e^{i\omega t}\mathbf{E}(\mathbf{k},\omega) = -\omega^2 e^{i\omega t}\ \mathbf{E}(\mathbf{k},\omega)$$

This gives us

$$\int_{\Re} d\omega\, e^{i\omega t}\int_{\Re^3} d^3k\,\left(-k^2\right)e^{i\,\mathbf{k}\cdot\mathbf{r}}\ \mathbf{E}(\mathbf{k},\omega) - \frac{1}{c^2}\int_{\Re} d\omega\,\left(-\omega^2\right)e^{i\omega t}\int_{\Re^3} d^3k e^{i\,\mathbf{k}\cdot\mathbf{r}}\ \mathbf{E}(\mathbf{k},\omega)$$
$$= \frac{1}{\varepsilon_0 c^2}\int_{\Re} d\omega\, e^{i\omega t}\left(-\omega^2\right)\int_{\Re^3} d^3k e^{i\,\mathbf{k}\cdot\mathbf{r}}\ \mathbf{P}(\mathbf{k},\omega)$$

But this equation can hold only if

$$\boxed{k^2\mathbf{E}(\mathbf{k},\omega) - \frac{\omega^2}{c^2}\mathbf{E}(\mathbf{k},\omega) = \frac{\omega^2}{\varepsilon_0 c^2}\mathbf{P}(\mathbf{k},\omega)}$$

This is the wave equation in the Fourier domain. It is just as complete a description as the wave equation in real space.

Note: In the Fourier domain the wave equation is purely algebraic – it contains no derivatives or integrals.

Note: In the absence of a polarizable medium, $\mathbf{P} \equiv 0$, the wave equation is local in Fourier space, i. e. the field at $\mathbf{k}, \omega$ does not depend on the field at $\mathbf{k}', \omega'$. This holds also for isotropic media with a linear response. medium is linear but non-isotropic, e. g., an isolated dipole in va, then light can be scattered from one plane wave into another with the same frequency but different wave vector, $\omega = \omega'$ and $\mathbf{k} \neq \mathbf{k}'$. If the response is nonlinear then $\mathbf{P}$ can lead to nonlinear wave mixing between fields with different frequency.

Note: In the presence of a polarization density $\mathbf{P} \neq 0$, the relationship between $|\mathbf{k}|$ and ω - the dispersion relation – is not as simple as in vacuum and must be worked out based on a microscopic theory of the medium response, i. e. we must find the $\mathbf{P}(\mathbf{k}, \omega)$ that results from a given $\mathbf{E}(\mathbf{k}, \omega)$.

Note: The Fourier representation of a plane wave is

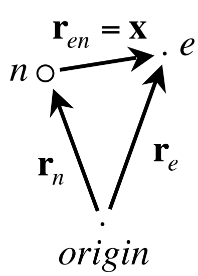
$$\mathbf{E}(\mathbf{k}, \omega) = \hat{\varepsilon} E_0 \delta(\mathbf{k}_0 - \mathbf{k}) \delta(\omega_0 - \omega)$$

In this course we will focus mostly on plane waves and their close cousins, Gaussian beams and wavepackets.

Theory of Atomic Response

So far we have developed a model for the electromagnetic field. Next, we need a model of how the microscopic constituents of the medium – the individual atoms – respond to the field. This will allow us to find the polarization density $\mathbf{P}$ as a function of the electric field $\mathbf{E}$.

Classical Atom: Simple model: a nucleus and a single electron



Equation of motion in external field

Lorentz force

$$\boxed{\mathbf{F} = q\left(\mathbf{E} + \underbrace{\mathbf{v}\times\mathbf{B}}_{\sim 0 \text{ if non-relativistic}}\right)}$$

Newton:

$$\boxed{\begin{aligned} &\text{(i)} \qquad m_e \frac{d^2}{dt^2}\mathbf{r}_e = e\mathbf{E}(\mathbf{r}_e, t) + \mathbf{F}_{en}(\mathbf{r}_{en}) \\ &\text{(ii)} \qquad m_n \frac{d^2}{dt^2}\mathbf{r}_n = -e\mathbf{E}(\mathbf{r}_n, t) - \mathbf{F}_{en}(\mathbf{r}_{en}) \end{aligned}}$$

As usual for the two-body problem, we can rewrite these equations on a form that provides better physical understanding. We define new variables,

$$\mathbf{x} = \mathbf{r}_{en} = \mathbf{r}_e - \mathbf{r}_n, \qquad m = \frac{m_e m_n}{m_e + m_n} \sim m_e,$$

$$\mathbf{R} = \frac{m_e\mathbf{r}_e + m_n\mathbf{r}_n}{M}, \qquad M = m_e + m_n \sim m_n,$$

where

$\mathbf{x}$: relative coordinate $\qquad m$: reduced mass

$\mathbf{R}$: center of mass coordinate $\qquad M$: total mass

We substitute these in the equations of motion, which can then be cast in the form

$$\boxed{\begin{aligned} M\frac{d^2}{dt^2}\mathbf{R} &= e\left[\mathbf{E}\left(\mathbf{R}+\frac{m_n}{M}\mathbf{x},t\right)-\mathbf{E}\left(\mathbf{R}-\frac{m_e}{M}\mathbf{x},t\right)\right] \\ m\frac{d^2}{dt^2}\mathbf{x} &= \frac{e}{2}\left[\mathbf{E}\left(\mathbf{R}+\frac{m_n}{M}\mathbf{x},t\right)+\mathbf{E}\left(\mathbf{R}-\frac{m_e}{M}\mathbf{x},t\right)\right]+\mathbf{F}_{en}(\mathbf{x})+\frac{1}{2}(m_n-m_e)\frac{d^2}{dt^2}\mathbf{R} \end{aligned}}$$

This is a <u>basic result with no approximations!</u>

In Milonni & Eberly, main body of text they $\begin{cases} \text{set} \quad \mathbf{R}\approx\mathbf{r}_n,\ \mathbf{x}\approx\mathbf{r}_{en} \\ \text{throw away the eq. for } \mathbf{R} \end{cases}$

We can do better, and
- clarify the approximations that we will make
- explore the consequences of the C.O.M. equation

<u>The Electric Dipole Approximation</u>

This is a key element in almost all of Optical Physics and Quantum Optics.

$$\boxed{\text{atomic dimensions } |\mathbf{x}|\sim 1\,\text{Å} \ll \text{optical wavelengths } \lambda\sim 1\mu\text{m}=10^4\,\text{Å}}$$

This implies that the electric field is nearly constant over the extent of an atom.

In this situation it is a good approximation to do a first order expansion in **x**:

$$\mathbf{E}\left(\mathbf{R}-\frac{m_e}{M}\mathbf{x},t\right)\approx\mathbf{E}(\mathbf{R},t)-\frac{m_e}{M}(\mathbf{x}\cdot\nabla)\mathbf{E}(\mathbf{R},t)$$

$$\mathbf{E}\left(\mathbf{R}+\frac{m_n}{M}\mathbf{x},t\right)\approx\mathbf{E}(\mathbf{R},t)+\frac{m_n}{M}(\mathbf{x}\cdot\nabla)\mathbf{E}(\mathbf{R},t)$$

Note: $(\vec{x}\cdot\nabla) = \begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{pmatrix}\cdot\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}$

where ε_i is the i'th cartesian component of $\vec{x}$

If we plug these into the equations for **R**, **x** we get

$$M\frac{d^2}{dt^2}\mathbf{R}\approx e(\mathbf{x}\cdot\nabla)\mathbf{E}(\mathbf{R},t) \qquad \text{Center-Of-Mass}$$

$$m\frac{d^2}{dt^2}\mathbf{x}=e\mathbf{E}(\mathbf{R},t)+\underbrace{\frac{m_n-m_e}{M}e(\mathbf{x}\cdot\nabla)\mathbf{E}(\mathbf{R},t)}_{\text{1st order in } \vec{x} \text{ (small)}}+\mathbf{F}_{en}(\mathbf{x}) \qquad \text{Rel. Coordinate}$$

Physical Interpretation

$\mathbf{p}=e\mathbf{x}$ is the electric dipole moment of the atom

The Center-Of-Mass equation can be written on the form

$$M\frac{d^2}{dt^2}\mathbf{R}\approx(\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{R},t)=\mathbf{F}=-\nabla_{\mathbf{R}}V(\mathbf{x},\mathbf{R},t)$$

F: Dipole Force

$V(\mathbf{x},\mathbf{R},t)=-\mathbf{p}\cdot\mathbf{E}(\mathbf{R},t)$: Interaction Energy

I. e. the Center-Of-Mass motion is governed by the Dipole Force, which is the gradient of the interaction energy between the dipole and the field.

The Relative Coordinate equation can be simplified by keeping only the leading terms in **x**,

$$m\frac{d^2}{dt^2}\mathbf{x} = e\mathbf{E}(\mathbf{R},t) + \mathbf{F}_{en}(\mathbf{x}) = -\nabla_{\mathbf{x}}V(\mathbf{x},\mathbf{R},t) + \mathbf{F}_{en}(\mathbf{x})$$

Pulling it all together, we get our final result

$M\frac{d^2}{dt^2}\mathbf{R} = -\nabla_{\mathbf{R}}V(\mathbf{x},\mathbf{R},t)$	Center-Of-Mass
$m\frac{d^2}{dt^2}\mathbf{x} = -\nabla_{\mathbf{x}}V(\mathbf{x},\mathbf{R},t) + \mathbf{F}_{en}(\mathbf{x})$	Rel. Coordinate
$V(\mathbf{x},\mathbf{R},t) = -\mathbf{p}\cdot\mathbf{E}(\mathbf{R},t)$	Electric Dipole Interaction

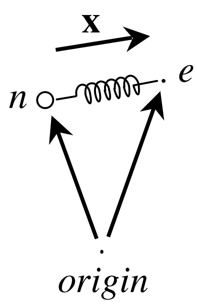
Note:

- The Center-Of-Mass equation is the foundation for a lot of important experimental techniques in the area of laser cooling and trapping. not explore this further during lectures in this course, but may rev homework. A more extensive discussion can be found in Steck's online notes.

- The equation for the internal degree of freedom allows us to find **p** as a function of **E**, and is thus the foundation for our theory of electromagnetic wave propagation in polarizable media.

The Electron Oscillator/Lorentz Atom

Consider a simple model of a classical atom, in which the electron is harmonically bound to the nucleus



$$\mathbf{F}_{en} = -m\omega_0^2\mathbf{x}$$

resonance frequency

Note: We should regard this as a model of the response of an atom, rather than a classical model of the atom itself.

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Now substitute the harmonic restoring force $\mathbf{F}_{en} = -m\omega_0^2 \mathbf{x}$ into the equation of motion for $\mathbf{x}$ to find

$$\boxed{\frac{d^2}{dt^2}\mathbf{x} + \omega_0^2 \mathbf{x} = \frac{e}{m}\mathbf{E}(\mathbf{R},t)}$$

We can combine this with $\boxed{\mathbf{P} = N\mathbf{p}, \quad \mathbf{p} = e\mathbf{x}}$

where N is the Number Density of atoms. This relates the macroscopic quantity $\mathbf{P}$ to the microscopic quantity $\mathbf{x}$

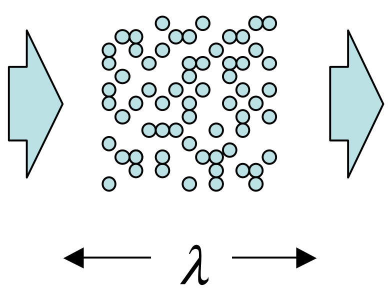
We now have

Maxwell's equations Lorentz model	$\Rightarrow$	Maxwell-Lorentz equations - We can seek self-consistent solutions to wave propagation

Later in the course we will replace the classical model of the atom with a full quantum model, which will lead to the analogous Maxwell-Bloch equations.

Classical Model of Absorption

Maxwell's equations predict than an oscillating dipole will loose energy by radiating an electromagnetic field. It is therefore necessary to refine our electron oscillator model to include damping.



Destructive interference in all directions except forward.

The Lorentz Model: - add an ad hoc friction term w/ $\beta << \omega_0$ (sub- critical damping)

$$\boxed{\frac{d^2}{dt^2}\mathbf{x} + 2\beta\frac{d}{dt}\mathbf{x} + \omega_0^2\mathbf{x} = \frac{e}{m}\mathbf{E}(\mathbf{R},t)}$$

- this is our basic equation for atomic response

Solution of the Equation of Motion for x:

Homework: Solve the equation of motion for **x** with

Driving field $\mathbf{E}(\mathbf{R},t) = \hat{\varepsilon} E_0 e^{-i(\omega t - kz)}$,

Trial solution $\mathbf{x}(\mathbf{R},t) = \mathbf{a}\, e^{-i(\omega t - kz)}$,

where **a** is a complex amplitude. Show that

$$\boxed{\mathbf{a} = -\hat{\varepsilon} \frac{(e/m)E_0}{\omega^2 - \omega_0^2 + 2i\beta\omega}}$$

Physical quantities:

$$\mathrm{Re}\left[\mathbf{E}(\mathbf{R},t)\right] = \hat{\varepsilon}E_0 \cos(\omega t - kz)$$

$$\mathrm{Re}\left[\mathbf{p}(\mathbf{R},t)\right] = \mathrm{Re}\left[e\mathbf{x}(\mathbf{R},t)\right] = \hat{\varepsilon}E_0 \frac{e^2}{m} \frac{\left(\omega_0^2 - \omega^2\right)\cos(\omega t - kz) + 2\beta\omega \sin(\omega t - kz)}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}$$

Note: Here we have assumed that the polarization vector $\hat{\varepsilon}$ is real, i. e. that the electromagnetic field is linearly polarized.

Note: **p** and **E** generally oscillate out of phase

$$\left.\begin{array}{lll} \omega \ll \omega_0 & \Rightarrow & \mathbf{p}\ \&\ \mathbf{E} \text{ in-phase} \\ \omega = \omega_0 & \Rightarrow & \mathbf{p} \text{ lags } \mathbf{E} \text{ by } \pi/2 \\ \omega \gg \omega_0 & \Rightarrow & \mathbf{p} \text{ lags } \mathbf{E} \text{ by } \pi \end{array}\right\} \text{ best to stick with complex notation!}$$

Complex Polarizability

We define the complex polarizability $\alpha(\omega)$ as follows

$$\boxed{\begin{gathered} \mathbf{p} = e\mathbf{x} = e\mathbf{a}e^{-i(\omega t - kz)} \equiv \alpha(\omega)\hat{\varepsilon}E_0 e^{-i(\omega t - kz)} \\ \alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} = \frac{e^2}{m}\frac{\omega_0^2 - \omega^2 + 2i\beta\omega}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2} \end{gathered}}$$

It is then easy to show that if $\mathbf{E}(\mathbf{r},t) = \hat{\varepsilon}E_0 e^{-i(\omega t - kz)}$ and $\mathbf{P} = N\mathbf{p}$, then the wave equation reduces to

$$\left(-k^2+\frac{\omega^2}{c^2}\right)\hat{\varepsilon}E_0 e^{-i(\omega t-kz)}=-\frac{\omega^2}{c^2}\frac{N\alpha(\omega)}{\varepsilon_0}\,\hat{\varepsilon}E_0 e^{-i(\omega t-kz)}$$

We thus have plane wave solutions, with a dispersion relation $k \equiv n(\omega)\omega/c$ that obeys

$$\boxed{k^2=\frac{\omega^2}{c^2}\left[1+\frac{N\alpha(\omega)}{\varepsilon_0}\right]\equiv\frac{\omega^2}{c^2}n^2(\omega)}$$

Here $n(\omega)$ is the complex index of refraction.

Switching into the Fourier domain, we can introduce the complex susceptibility

$$\mathbf{P}(\omega)=\varepsilon_0\chi(\omega)\mathbf{E}(\omega), \qquad \chi(\omega)\equiv\frac{N\alpha(\omega)}{\varepsilon_0}.$$

Recalling that $\mathbf{E}(\omega)\equiv FT\left[\mathbf{E}(t)\right]$ and $\mathbf{P}(t)=\varepsilon_0\int_{-\infty}^{t}R(t-t')\mathbf{E}(t')dt'$, and that convolution in the time domain corresponds to multiplication in the frequency domain, we see that the complex susceptibility is related to the linear response function as

$$\chi(\omega)\equiv FT\left[R(\tau)\right].$$

Thus we have

time domain description		frequency domain description
$\mathbf{P}(t),\quad R(\tau)$	$\xleftrightarrow{\text{FT}}$	$\mathbf{P}(\omega),\quad \chi(\omega)$

Complex Index of Refraction

Let $$\boxed{n(\omega)=n_R(\omega)+i\,n_I(\omega)}$$

and consider the propagation of a plane wave

$$\mathbf{E}(z,t)=\hat{\varepsilon}E_0 e^{-i(\omega t-kz)}=\hat{\varepsilon}E_0 e^{-i\left(\omega t-[n(\omega)\omega/c]z\right)}=\hat{\varepsilon}E_0 e^{-n_I(\omega)\omega z/c}e^{-i\omega\left(t-n_R(\omega)z/c\right)}$$

We can now identify

$$\frac{c}{\omega n_I(\omega)} \longleftarrow \text{attenuation length}$$

$$\frac{c}{n_R(\omega)} \longleftarrow \text{phase velocity}$$

Absorption

The intensity of the plane wave electromagnetic field **E** is

$$\mathrm{I}_\omega(z)=\tfrac{1}{2}n_R(\omega)c\varepsilon_0\left|E(z,t)\right|^2=\mathrm{I}_\omega(0)e^{-2n_I(\omega)\omega z/c}\equiv I_0 e^{-a(\omega)z},$$

where the absorption or extinction coefficient is given by

$$a(\omega)\equiv 2n_I(\omega)\omega/c=\frac{2\omega}{c}\mathrm{Im}\left[\left(1+\frac{N\alpha(\omega)}{\varepsilon_0}\right)^{1/2}\right]$$

Possibility of Gain?

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