Quantum States of the Quantized Field

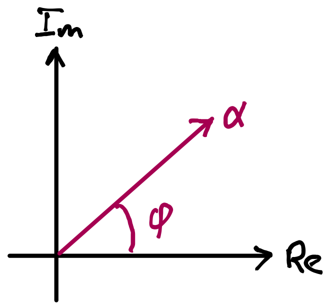
Amplitude and Phase

– Key characteristics of classical fields
– Need equivalents for quantum fields

Classical Field

$$E(z,t) = \mathcal{E}_k \boxed{\alpha} e^{-i(\omega t - kz)} + c.c.$$

$|\alpha| e^{i\varphi}$



Quantum Field

$$\hat{E}(z,t) = \mathcal{E}_k \boxed{\hat{a}} e^{-i(\omega t - kz)} + H.C.$$

Non-Hermitian!
Separate in amplitude & phase?

Consider operators

$$\hat{a} = (\hat{N}+1)^{1/2} \hat{\exp}(i\varphi)$$
$$\hat{a}^{+} = \hat{\exp}(-i\varphi)(\hat{N}+1)^{1/2}$$

"phase" "amplitude"

$$\hat{\exp}(i\varphi) = (\hat{N}+1)^{-1/2} \hat{a}$$
$$\hat{\exp}(-i\varphi) = \hat{a}^{+}(\hat{N}+1)^{-1/2}$$

"Phase operators"

$$\hat{\exp}(i\varphi)\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$
$$\hat{\exp}(-i\varphi)\hat{\exp}(i\varphi) = 1 \qquad = \left[\hat{\exp}(-i\varphi)\right]^{-1}$$

– Analogous to classical phases
– Non-Hermitian, NOT observables

Quadrature operators?

$$\hat{\cos}\varphi = \frac{1}{2}\left[\hat{\exp}(i\varphi) + \hat{\exp}(-i\varphi)\right]$$
$$= \frac{1}{2}\left[(\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

$$\hat{\sin}\varphi = \frac{1}{2i}\left[\hat{\exp}(i\varphi) - \hat{\exp}(i\varphi)\right]$$
$$= \frac{1}{2i}\left[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

– Hermitian -> observables
– but ultimately too cumbersome

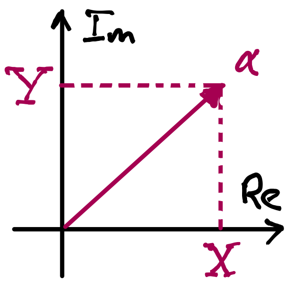
Let's rewind and try again…

Quantum States of the Quantized Field

Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k \alpha_k(t)}\, e^{ikz} + c.c.$$

complex amplitude for mode e^{ikz}



Define

$$\boxed{\begin{aligned} X(t) &= \mathrm{Re}[\alpha_k(t)] = \tfrac{1}{2}[\alpha_k(t) + \alpha_k^*(t)] = Q(t) \\ Y(t) &= \mathrm{Im}[\alpha_k(t)] = \tfrac{1}{2i}[\alpha_k(t) - \alpha_k^*(t)] = P(t) \end{aligned}}$$

Quantization: $\alpha \to \hat{a}, \quad \alpha^* \to \hat{a}^+$

$$\left.\begin{aligned} \hat{X}(t) &= \tfrac{1}{2}[\hat{a}_k(t) + \hat{a}_k^+(t)] = \hat{Q}(t) \\ \hat{Y}(t) &= \tfrac{1}{2i}[\hat{a}_k(t) - \hat{a}_k^+(t)] = \hat{P}(t) \end{aligned}\right\} \quad [\hat{X}(t), \hat{Y}(t)] = i/2$$

$$\begin{aligned} \hat{E}(z,t) &= \mathcal{E}_k(\hat{X}(t) + i\hat{Y}(t))e^{ikz} + H.C. \\ &= \mathcal{E}_k[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)] \end{aligned}$$

– same info, easier to work with –

Quantum States of the Field in Mode k

Number States (Foch states)

$$\boxed{\hat{a}^+\hat{a}|n\rangle = n|n\rangle}$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \tfrac{1}{2}(n + 1/2)$$

$$\boxed{\Delta X \Delta Y = \tfrac{1}{2}(n + 1/2)}$$

– HIGHLY non-classical, $\langle \hat{E} \rangle = 0$

– VERY hard to make for large n

Quantum States of the Quantized Field

Coherent States (Quasi-classical states)

– Closest approximation to classical field

– See Cohen-Tannoudji, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle\hat{X}(t)\rangle = \langle\psi|\hat{X}(t)|\psi\rangle = X(t), \quad \langle\hat{Y}(t)\rangle = Y(t)$$

$$\langle\hat{H}(t)\rangle = \hbar\omega\left(|\alpha(t)|^2 + 1/2\right)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^+(t) = \hat{a}^+(0)e^{i\omega t}$$

equivalently

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

(1) $\langle\hat{a}(0)\rangle = \langle\psi|\hat{a}(0)|\psi\rangle = \alpha(0)$

(2) $\langle\hat{a}^+(0)\hat{a}(0)\rangle = \alpha(0)^*\alpha(0)$

Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

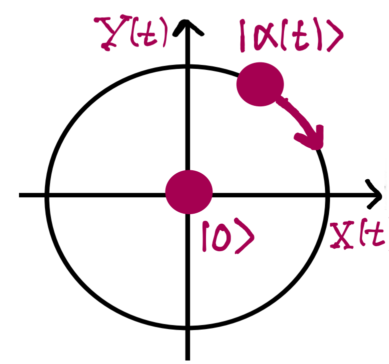
Physical properties

$$\langle\hat{X}(t)\rangle = \mathrm{Re}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\langle\hat{Y}(t)\rangle = \mathrm{Im}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$

$$\Delta X\,\Delta Y = 1/4$$



Quantum States of the Quantized Field

Begin Lecture 04-12-2021

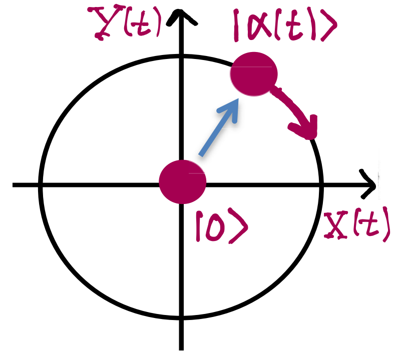
Photon statistics

Measure $\hat{N}$ ⇒ $\begin{cases} \text{outcomes } n \\ P(n) = \langle\alpha|n\rangle\langle n|\alpha\rangle = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \end{cases}$

Poisson distribution w/ $\begin{cases} \text{mean} \quad \bar{n} = |\alpha|^2 \\ \text{variance} \quad \Delta n^2 = |\alpha|^2 \end{cases}$

$$\boxed{\Delta n = \sqrt{\bar{n}} \quad \text{– Shot Noise}}$$

More about Coherent States



Coherent States as translated Vacuum States?

Generating Coherent States from the Vacuum

Definition: $\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$

Unitary, equals translation

Glaubers formula (from BCH formula)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Quantum States of the Quantized Field

Apply to $[\underset{\hat{A}}{\underset{\uparrow}{\alpha\hat{a}^{+}}}, \underset{\hat{B}}{\underset{\uparrow}{-\alpha^{*}\hat{a}}}] = \underset{[\hat{A},\hat{B}]}{\underset{\uparrow}{\alpha^{*}\alpha}}$

$$\boxed{\hat{D}(\alpha) = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^{+}}\, e^{-\alpha\hat{a}}}$$

Remember: $\hat{a}|0\rangle = 0 \Rightarrow$

$$e^{-\alpha^{*}\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^{*}\hat{a})^n}{n!}|0\rangle = |0\rangle$$

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^{+}}|0\rangle$$

$$= e^{-|\alpha|^2}\sum_n \frac{(\alpha\hat{a}^{+})^n}{n!}|0\rangle$$

$$= e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

$$\boxed{\hat{D}(\alpha)|0\rangle = |\alpha\rangle}$$

OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat{a}^{+} - \alpha^{*}\hat{a} = (\alpha-\alpha^{*})\hat{X} + i(\alpha+\alpha^{*})\hat{Y}$$

$$= i2Y\hat{X} + i2X\hat{Y}$$

where $X = \langle\alpha|\hat{X}|\alpha\rangle$, $Y = \langle\alpha|\hat{Y}|\alpha\rangle$

Glaubers formula again:

$$\boxed{\hat{D}(\alpha) = e^{i2Y\hat{X} + i2X\hat{Y}} = e^{-XY/4}\, e^{i2Y\hat{X}}\, e^{i2X\hat{Y}}}$$

Recall: $\hat{S}(q) = e^{-iq\hat{p}/\hbar} \Rightarrow$ **translation by** q

$\hat{S}(p) = e^{-ip\hat{q}/\hbar} \Rightarrow$ **translation by** p

where $q = q_0 X$, $p = p_0 Y$

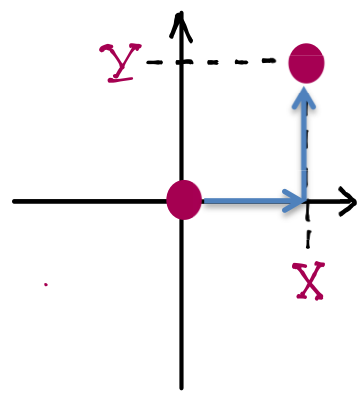
$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$ & $x_0 p_0 = 2\hbar$

Quantum States of the Quantized Field

This gives us

$$\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}$$

$\hat{D}(\alpha)$ translates along X then Y



Discussion –
How to do this?

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t)$ @ $t = 0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k (\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\hat{H} = \hbar\omega_k(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a} + \hat{a}^+)$$

$$\lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)$$

Homework Problem (voluntary)

$$\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega_k)T/2} \frac{\sin[(\omega-\omega_k)T/2]}{(\omega-\omega_k)/2}$$

Quantum States of the Quantized Field

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Homework Problem (voluntary)

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Recall from Semi-Classical Laser Theory

$\langle\hat{\mu}(t)\rangle$ drives $\hat{E}(t)$

classical dipole + quantum fluctuations

coherent state + quantum fluctuations

Drive from $0 < t < T$

$$\alpha(t) = \alpha(T) e^{-i\omega_k(t-T)}$$

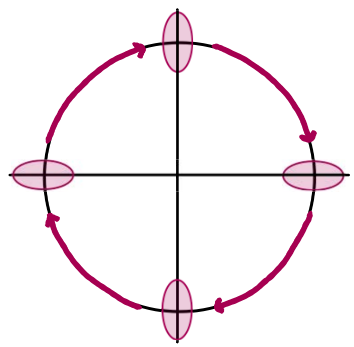
Quantum States of the Quantized Field

Squeezed States

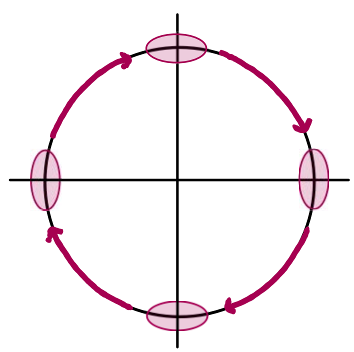
Minimum uncertainty states w/asymmetry

$$\Delta X \Delta Y = 1/4\,, \quad \Delta X(t) \neq \Delta Y(t)$$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium

Quantum States of the Quantized Field

Odds and Ends – Thermal States

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_B T}\right]$$

$$\hat{\rho} = \sum_n P(n)|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-E_n/k_B T}|n\rangle\langle n|$$

$$= (1-q)\sum_n q^n |n\rangle\langle n|, \quad q = e^{-\hbar\omega/k_B T}$$

Mean Photon Number:

$$\bar{n} = \mathrm{Tr}(\hat{\rho}\hat{N}) = \sum_{m,n}\langle m|(1-q)q^n|n\rangle\langle n|\hat{N}|m\rangle$$

$$= (1-q)\sum_n n q^n = \frac{q}{1-q}$$

Photon Number Uncertainty:

$$\langle \hat{N}^2\rangle = (1-q)\sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

$$\Delta n^2 = \langle \hat{N}^2\rangle - \langle \hat{N}\rangle^2$$

$$= \frac{q^2+q}{(1-q)} - \frac{q^2}{(1-q)^2} = \frac{q}{(1-q)^2}$$

$$\bar{n} = \frac{q}{1-q}$$

Coherent State limit

$$\Delta n = \frac{\sqrt{q}}{1-q} = \sqrt{\bar{n}(\bar{n}+1)} \geq \sqrt{\bar{n}}$$

Optical Frequencies, Room Temperature:

$\lambda = 1\,\mu m, \quad T = 300\,K$

$q = 6.5\times 10^{-6}, \quad \bar{n} \sim 10^{-6}$

Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

Use $[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$ **to show**

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t}\hat{T}_\alpha$$

$\Rightarrow$ $\hat{T}_\alpha\hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$

$\Rightarrow$ $\hat{T}_\alpha\hat{a}\hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$

From this we get

$$\hat{E}_\perp' = \hat{T}_\alpha\hat{E}_\perp\hat{T}_\alpha^+ = \hat{T}_\alpha\left(\mathcal{E}_k\hat{a}e^{i\vec{k}\cdot\vec{r}} + H.C.\right)\hat{T}_\alpha^+$$

$$= \mathcal{E}_k\hat{a}e^{i\vec{k}\cdot\vec{r}} + H.C. + \mathcal{E}_k\alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

$$= \hat{E}_\perp + \hat{E}_\perp^{Cl}(\alpha, t)$$

We also have $|\psi'(t)\rangle = \hat{T}_\alpha|\alpha(t)\rangle = |0\rangle$

Action of the unitary transformation $\hat{T}_\alpha(t)$

$$\hat{E}_\perp' = \hat{T}_\alpha(t)\hat{E}_\perp\hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

$$|\psi'(t)\rangle = T_\alpha(t)|\alpha(t)\rangle = |0\rangle$$

$\Downarrow$

We can work with

$\hat{E}_\perp$, $|\alpha(t)\rangle$ or $\hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$, $|0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!