Quantized Light – Matter Interactions

$$\hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+ \hat{a} e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+ \hat{a}^+ e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a} e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_- \hat{a}^+ e^{-i(\omega_{21}-\omega)t}\right)$$

RWA and resonant approximation

Begin 04-26-2021

Jaynes-Cummings Hamiltonian

$$\boxed{\hat{H}_I = \hat{H}_{AF} = \hbar g\left(\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^+ e^{-i\Delta t}\right)}$$

$\Delta = \omega_{21} - \omega$

Eigenstates of $\hat{H}_0 = \hat{H}_F + \hat{H}_A$

State	Energy
$\|2,n\rangle$	$\hbar\omega n + \hbar\omega_{21}$
$\|1,n+1\rangle$	$\hbar\omega(n+1) - \hbar\omega_{21}$

Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |2,n\rangle$$

$$|\psi(t)\rangle = C_{1,n+1}|1,n+1\rangle + C_{2,n}|2,n\rangle$$

Matrix elements

$$\langle 2,n|\hat{H}_{AF}|1,n+1\rangle = \hbar g\sqrt{n+1}\, e^{i\Delta t}$$

$$\langle 1,n+1|\hat{H}_{AF}|2,n\rangle = \hbar g\sqrt{n+1}\, e^{-i\Delta t}$$

Schrödinger Equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_{1,n+1} \\ C_{2,n}\end{pmatrix} = \hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0\end{pmatrix}\begin{pmatrix} C_{1,n+1} \\ C_{2,n}\end{pmatrix}$$

Quantized Light – Matter Interactions

Cavity QED version of the Rabi Problem

$$|\psi(0)\rangle = |2,n\rangle$$

$$|\psi(t)\rangle = C_{1,n+1}|1,n+1\rangle + C_{2,n}|2,n\rangle$$

Matrix elements

$$\langle 2,n|\hat{H}_{AF}|1,n+1\rangle = \hbar g\sqrt{n+1}\, e^{i\Delta t}$$

$$\langle 1,n+1|\hat{H}_{AF}|2,n\rangle = \hbar g\sqrt{n+1}\, e^{-i\Delta t}$$

Schrödinger Equation

$$i\hbar\frac{d}{dt}\begin{pmatrix} C_{1,n+1} \\ C_{2,n}\end{pmatrix} = \hbar g\sqrt{n+1}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0\end{pmatrix}\begin{pmatrix} C_{1,n+1} \\ C_{2,n}\end{pmatrix}$$

$$\boxed{\begin{aligned}\dot{C}_{1,n+1} &= -ig\sqrt{n+1}\, e^{-i\Delta t}\, C_{2,n} \\ \dot{C}_{2,n} &= -ig\sqrt{n+1}\, e^{i\Delta t}\, C_{1,n+1}\end{aligned}}$$

Substitute $C_{1,n+1} \rightarrow C_1$, $C_{2,n} \rightarrow C_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $C_1(0) = 0$, $C_2(0) = 1$

$$\boxed{\begin{aligned} C_{2,n}(t) &= \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] e^{i\Delta t/2} \\ C_{1,n+1} &= -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2} \\ \Omega_n &= \left(4g^2(n+1) + \Delta^2\right)^{1/2}\end{aligned}}$$

Quantized Light – Matter Interactions

$$\dot{c}_{1,n+1} = -ig\sqrt{n+1}\, e^{-i\Delta t}\, c_{2,n}$$

$$\dot{c}_{2,n} = -ig\sqrt{n+1}\, e^{i\Delta t}\, c_{1,n+1}$$

Substitute $c_{1,n+1} \rightarrow c_1 \,,\; c_{2,n} \rightarrow c_2 e^{i\Delta t}$

Looks exactly like Semiclassical Rabi problem

Solve for $c_1(0) = 0 \,,\; c_2(0) = 1$

$$c_{2,n}(t) = \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] e^{i\Delta t/2}$$

$$c_{1,n+1} = -i\,\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2}$$

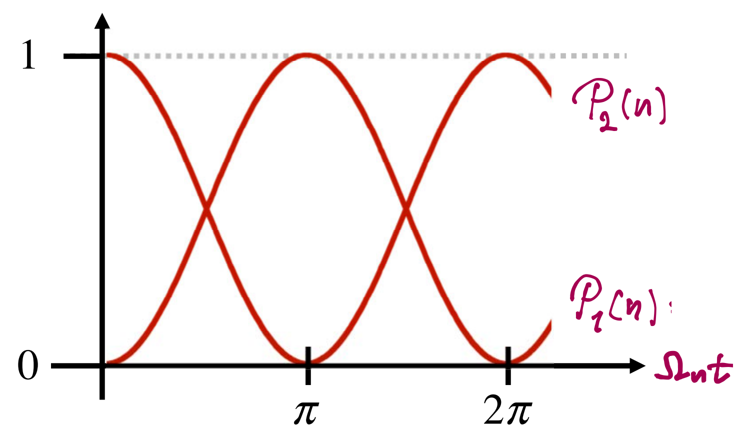
$$\Omega_n = \left(4g^2(n+1) + \Delta^2\right)^{1/2}$$

Rabi Oscillations

$$P_2(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$P_1(n) = \frac{4g^2(n+1)}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right)$$

Example: $\Delta = 0$



Quantized Light – Matter Interactions

Rabi Oscillations

$$P_2(n) = \cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$P_1(n) = \frac{4g^2(n+1)}{\Omega_n^2} \sin^2\left(\frac{\Omega_n t}{2}\right)$$

Example: $\Delta = 0$

Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |2,0\rangle$ ⇨ no photons in field

yet $|2,0\rangle$ evolves into $|1,1\rangle$

Uniquely QED phenomenon!

Asymmetry $\begin{cases} |2, n=0\rangle \longrightarrow |1, n=1\rangle \\ |1, n=0\rangle \longrightarrow |1, n=0\rangle \end{cases}$

holds germ of Spontaneous Decay

Quantized Light – Matter Interactions

Vacuum Rabi Oscillations

If $|\psi(0)\rangle = |2,0\rangle$ ⇒ no photons in field

yet $|2,0\rangle$ evolves into $|1,1\rangle$

Uniquely QED phenomenon!

Asymmetry $\begin{cases} |2,n=0\rangle \rightarrow |1,n=1\rangle \\ |1,n=0\rangle \rightarrow |1,n=1\rangle \end{cases}$

holds germ of **Spontaneous Decay**

More Cavity QED – Coherent States

2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

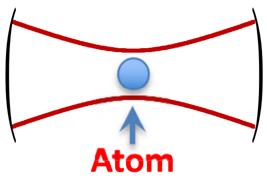
$$|\psi(0)\rangle = \underbrace{|1\rangle}_{\text{atom}} \otimes \underbrace{|\alpha\rangle}_{\text{field}} = \sum_n C_n |1,n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}$$

From Rabi solutions: $\Delta = 0$ ⇒

$$C_{1,n} = \cos\left(\frac{\Omega_n t}{2}\right), \quad C_{2,n-1} = -i\sin\left(\frac{\Omega_n t}{2}\right)$$

Quantized Light – Matter Interactions

More Cavity QED – Coherent States



2-level atom

Single cavity mode

What happens with a Coherent State in the Cavity mode?

(Quantum-Classical correspondence)

Initial atom-field state:

$$|\psi(0)\rangle = |1\rangle \otimes |\alpha\rangle = \sum_n C_n |1,n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

atom field

From Rabi solutions: $\Delta = 0$

$$C_{1,n} = \cos\left(\frac{\Omega_n t}{2}\right), \quad C_{2,n-1} = -i \sin\left(\frac{\Omega_n t}{2}\right)$$

Therefore

uncoupled

$$|\psi(t)\rangle = C_0 |1,0\rangle + \sum_{n=1}^{\infty} C_n \left[\cos\left(\frac{\Omega_n t}{2}\right)|1,n\rangle - i \sin\left(\frac{\Omega_n t}{2}\right)|2,n-1\rangle\right]$$

Consider the Atomic Excited State Population

$$P_2(t) = \sum_{n=0}^{\infty} P_{2,n} = \sum_{n=0}^{\infty} |\langle 2,n|\psi(t)\rangle|^2$$

$$= \sum_{n=0}^{\infty} |C_n|^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \sin^2\left(\frac{\Omega_n t}{2}\right)$$

Use $|\alpha|^2 = \bar{n}$ **and** $\Omega_n = 2g\sqrt{n}$

$$\boxed{P_2(t) = \sum_{n=0}^{\infty} \frac{(\bar{n})^n e^{-\bar{n}}}{n!} \sin^2(g\sqrt{n}\,t)}$$

Quantized Light – Matter Interactions

Therefore uncoupled

$$|\psi(t)\rangle = c_0|1,0\rangle + \sum_{n=1}^{\infty} c_n\left[\cos\left(\frac{\Omega_n t}{2}\right)|1,n\rangle - i\sin\left(\frac{\Omega_n t}{2}\right)|2,n-1\rangle\right]$$

Consider the Atomic Excited State Population

$$P_2(t) = \sum_{n=0}^{\infty} P_{2,n} = \sum_{n=0}^{\infty} |\langle 2,n|\psi(t)\rangle|^2$$

$$= \sum_{n=0}^{\infty} |c_n|^2 \sin^2\left(\frac{\Omega_n t}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \sin^2\left(\frac{\Omega_n t}{2}\right)$$

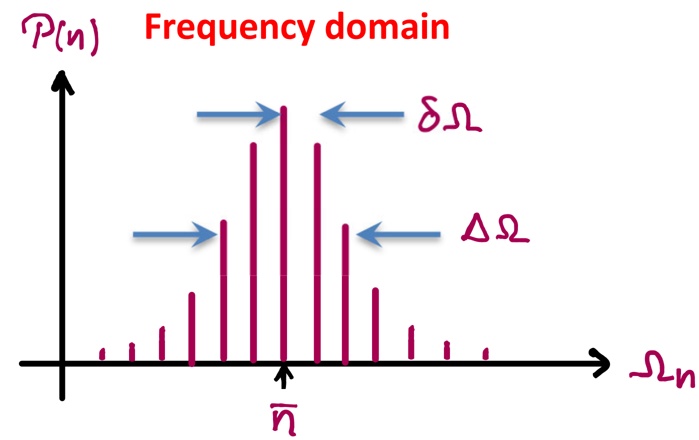
Use $|\alpha|^2 = \bar{n}$ and $\Omega_n = 2g\sqrt{n}$

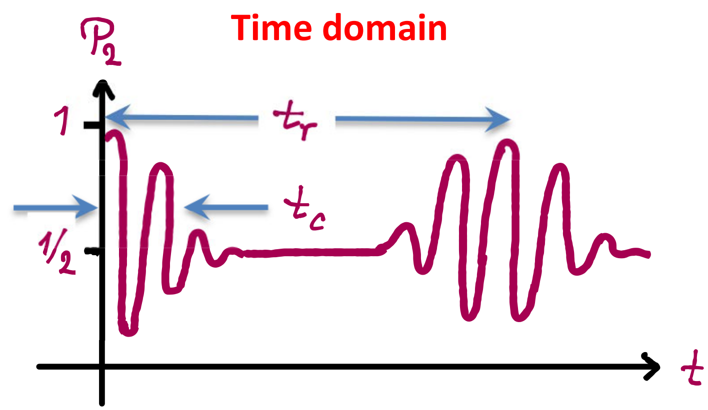
$$\boxed{P_2(t) = \sum_{n=0}^{\infty} \frac{(\bar{n})^n e^{-\bar{n}}}{n!} \sin^2(g\sqrt{n}\,t)}$$

– Poisson weighted average of sinusoids

– Sinusoids gradually dephase over time

Collapse of oscillation amplitude





Quantized Light – Matter Interactions

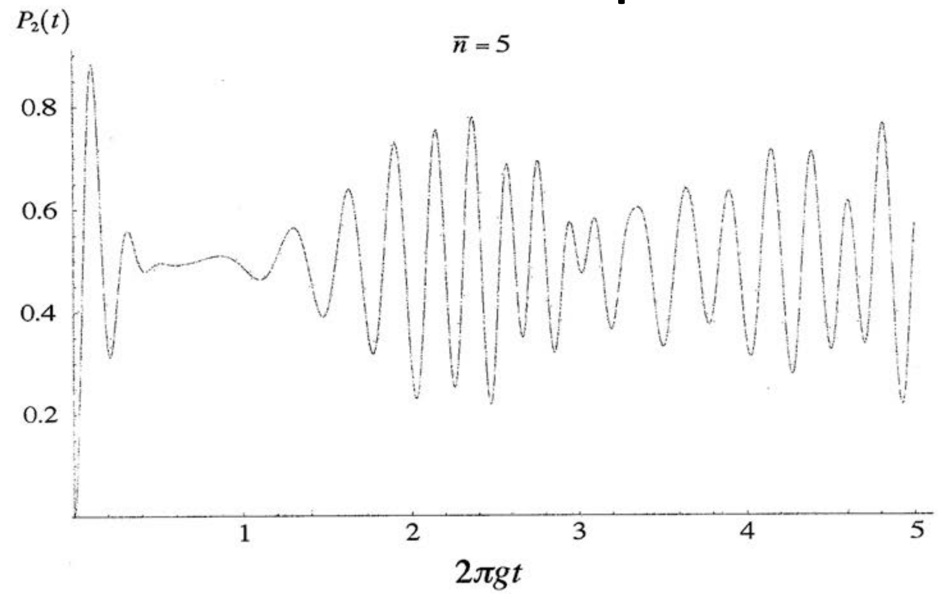
- Poisson weighted average of sinusoids
- Sinusoids gradually dephase over time

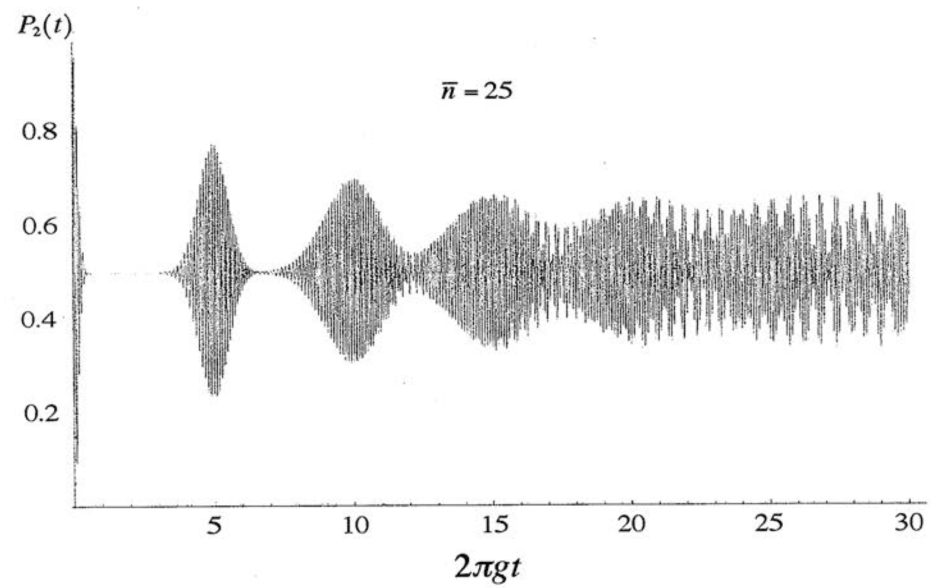
Collapse of oscillation amplitude

Frequency domain

Time domain

Numerical examples



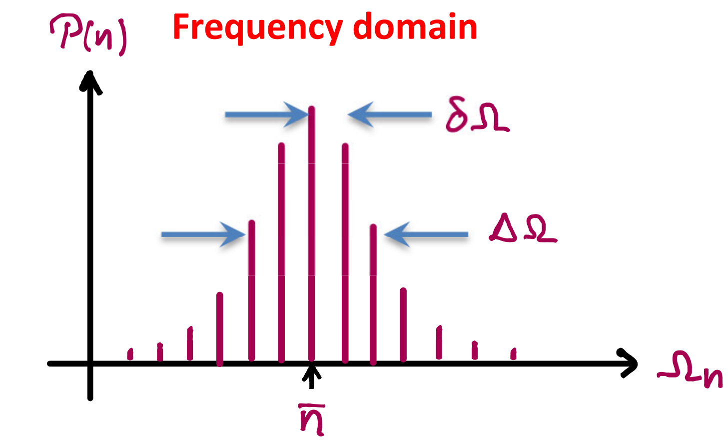


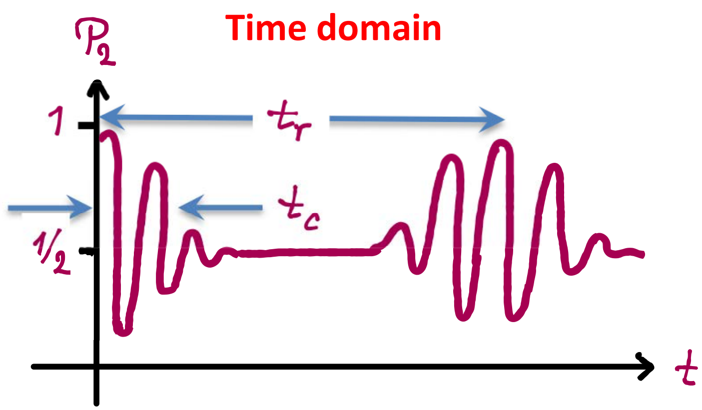
Quantized Light – Matter Interactions

– Poisson weighted average of sinusoids

– Sinusoids gradually dephase over time

Collapse of oscillation amplitude





Use $\Delta n = \sqrt{\bar{n}}$ ⇒ $\Delta\Omega \sim \Delta\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta\Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$\boxed{t_c = \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}}$$

for $\bar{n} \gg \sqrt{\bar{n}}$

Rephasing: when $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})\, t_r \simeq 2\pi m$

Similar arguments -> Revival time

$$\boxed{t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}}$$

Quantized Light – Matter Interactions

Use $\Delta n = \sqrt{\bar{n}}$ ⇨ $\Delta\Omega \sim \Delta\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Delta\Omega_{\bar{n}-\sqrt{\bar{n}}}$

$$\boxed{t_c = \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}}} \sim \frac{1}{2g}}$$

for $\bar{n} \gg \sqrt{\bar{n}}$

Rephasing: when $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})t_r \simeq 2\pi m$

Similar arguments -> Revival time

$$\boxed{t_r \sim \frac{2\pi}{\delta\Omega} \sim \frac{2\pi\sqrt{\bar{n}}}{g}}$$

Collapse & Revival Dynamics

Pure Quantum Phenomenon

("graininess" of photons)

Classical limit

$V \to \infty$ ⇨ $\mathcal{E}_k \to 0$ ⇨ $g \to 0$, $t_c \to \infty$

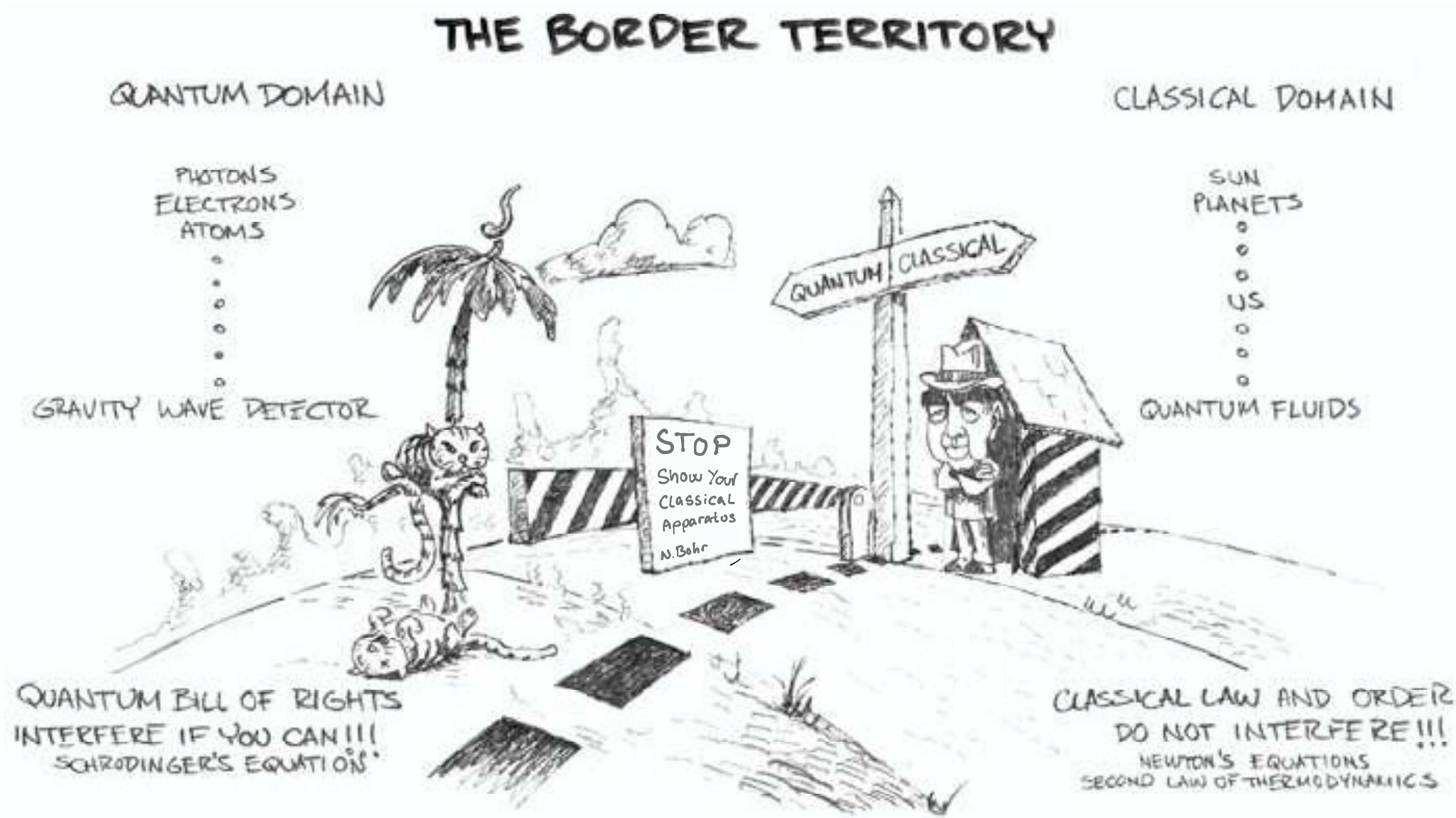
$\bar{n} \to \infty$ ⇨ $\frac{\Delta\Omega}{\Omega_{\bar{n}}} \to 0$ ⇨ $\Omega_{\bar{n}} > 0$ well defined

$$\boxed{\Omega_{\bar{n}} = 2g\sqrt{\bar{n}} = \frac{\vec{\mu}_{21}\cdot 2\vec{\mathcal{E}}_k \mathcal{E}_k\sqrt{\bar{n}}}{\hbar} = \frac{\vec{\mu}_{21}\cdot\vec{E}}{\hbar}}$$

Classical Rabi frequency

mean field $\langle\alpha(t)|\hat{\vec{E}}|\alpha(t)\rangle$

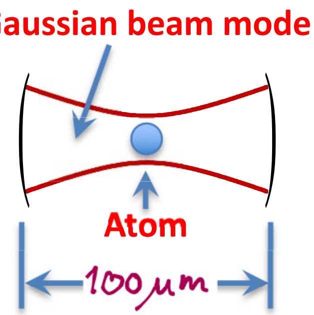
End 04-26-2021


THE BORDER TERRITORY
QUANTUM DOMAIN
CLASSICAL DOMAIN
PHOTONS
ELECTRONS
ATOMS
GRAVITY WAVE DETECTOR
SUN
PLANETS
US
QUANTUM FLUIDS
QUANTUM
CLASSICAL
STOP
Show Your
Classical
Apparatus
N. Bohr
QUANTUM BILL OF RIGHTS
INTERFERE IF YOU CAN!!!
SCHRODINGER'S EQUATIONS
CLASSICAL LAW AND ORDER
DO NOT INTERFERE!!!
NEWTON'S EQUATIONS
SECOND LAW OF THERMODYNAMICS

Quantized Light – Matter Interactions

Quantized Light – Matter Interactions

More Cavity QED – Dressed States



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

Energy levels of the atom-cavity system

Bare & Dressed States

Return to single - mode result

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

$$\hbar\omega \hat{a}^+\hat{a} + \frac{1}{2}\omega_{21}\hat{\sigma}_z \Rightarrow H_0$$

$$+\hbar g(\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{\sigma}_- \hat{a}^+ e^{-i\Delta t}) \Rightarrow H_{AF}$$

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$\|1,n\rangle$	$E_{1,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\|2,n-1\rangle$	$E_{2,n-1} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

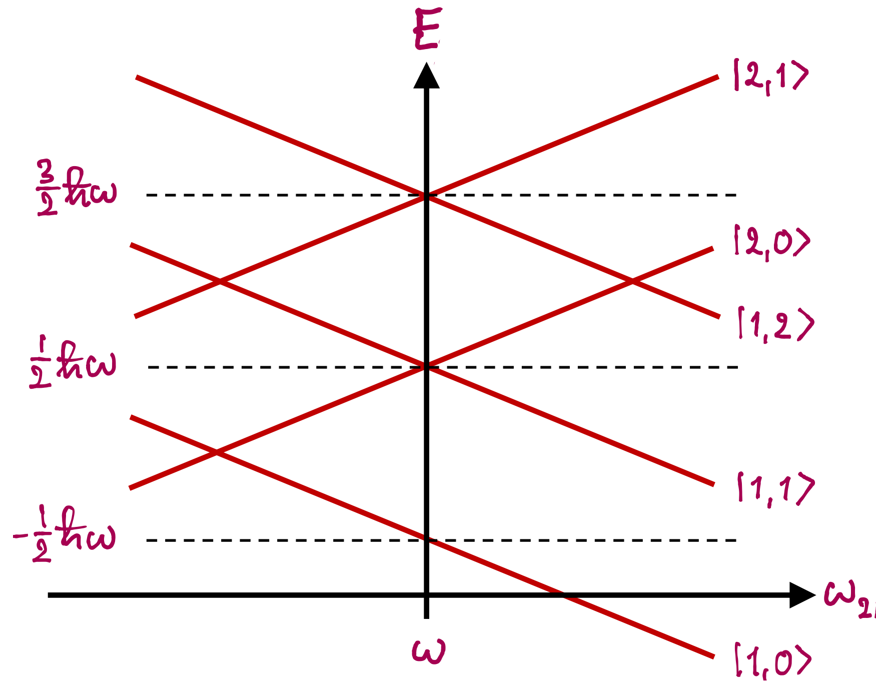
Quantized Light – Matter Interactions

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$\vert 1,n\rangle$	$E_{1,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\vert 2,n-1\rangle$	$E_{2,n-1} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

Imagine we can tune ω_{21}

Energy level diagram



Crossings @ $\omega = \omega_{21}$ are degeneracies of pairs with n shared excitations

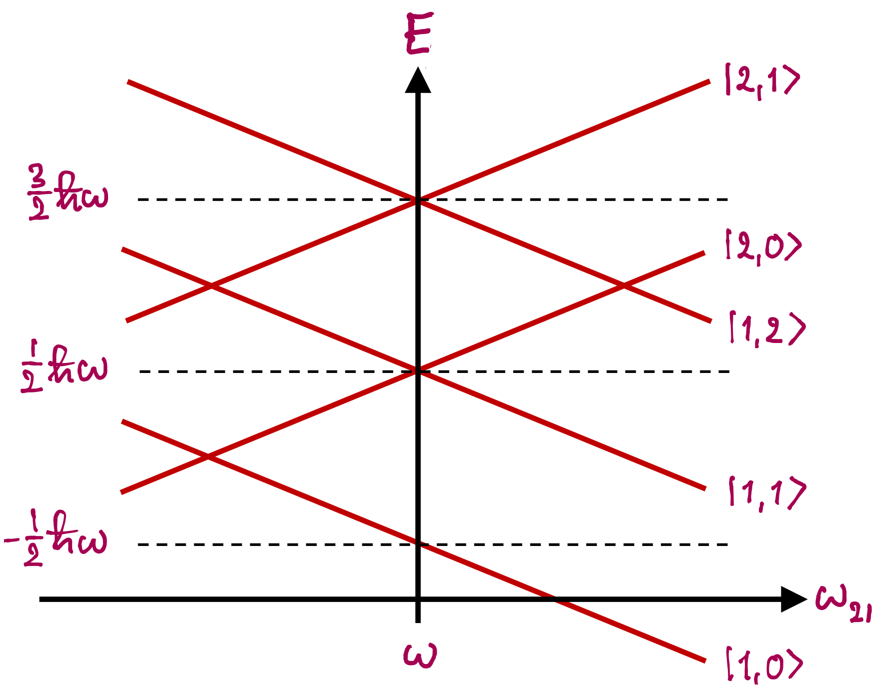
$$\left\{\begin{array}{ll} n=0 & |1,0\rangle \\ n=1 & \{|1,1\rangle, |2,0\rangle\} \\ n=2 & \{|1,2\rangle, |2,1\rangle\} \\ \vdots & \vdots \\ n & \{|1,n\rangle, |2,n-1\rangle\} \end{array}\right.$$

Quantized Light – Matter Interactions

"Bare" states ($g = 0$, eigenstates of H_0)

State	Energy
$\lvert 1, n \rangle$	$E_{1,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\lvert 2, n-1 \rangle$	$E_{2,n-1} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

Energy level diagram



"Dressed" states { eigenstates of $\hat{H} = \hat{H}_0 + \hat{H}_{AF}$

Structure of $\hat{H}$:

1x1 block

$$\hat{H} = \begin{bmatrix} \hat{H}_0 & & & \\ & \hat{H}_1 & & \\ & & \hat{H}_2 & \\ & & & \ddots \end{bmatrix}$$

2x2 blocks

Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n - \tfrac{1}{2})\hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

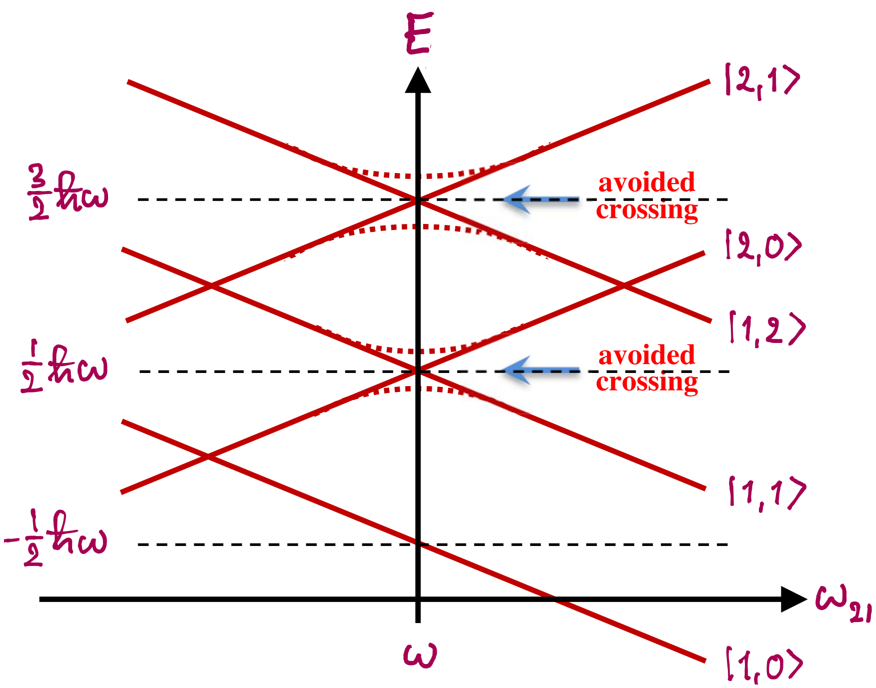
$\Delta = \omega_{21} - \omega$

Quantized Light – Matter Interactions

"Bare" states ($g=0$, eigenstates of H_0)

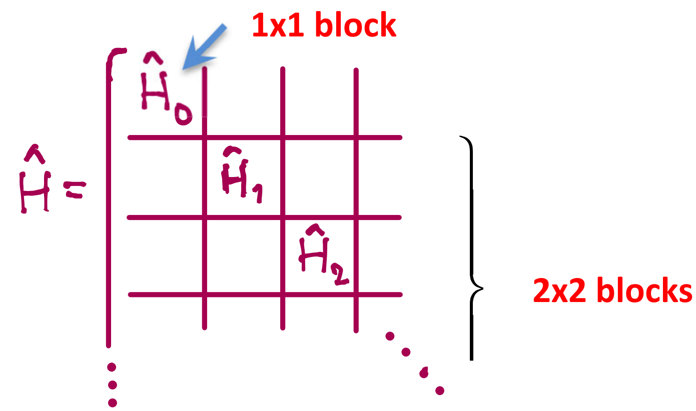
State	Energy
$\|1,n\rangle$	$E_{1,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\|2,n-1\rangle$	$E_{2,n-1} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

Energy level diagram



"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$:



Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n-\tfrac{1}{2})\hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

$\Delta = \omega_{21} - \omega$

Quantized Light – Matter Interactions

"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$:

1x1 block

$$\hat{H} = \begin{bmatrix} \hat{H}_0 & & & \\ & \hat{H}_1 & & \\ & & \hat{H}_2 & \\ \vdots & & & \ddots \end{bmatrix}$$

2x2 blocks

Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n - \tfrac{1}{2}) \hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

$\Delta = \omega_{21} - \omega$

Eigenvalues $E_\pm = (n - \tfrac{1}{2})\hbar\omega \pm \frac{\hbar}{2}\sqrt{4g^2 n + \Delta^2}$

Eigenstates

$$|+,n\rangle = \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(\theta_n/2\right)|1,n\rangle + \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left(\theta_n/2\right)|2,n-1\rangle$$

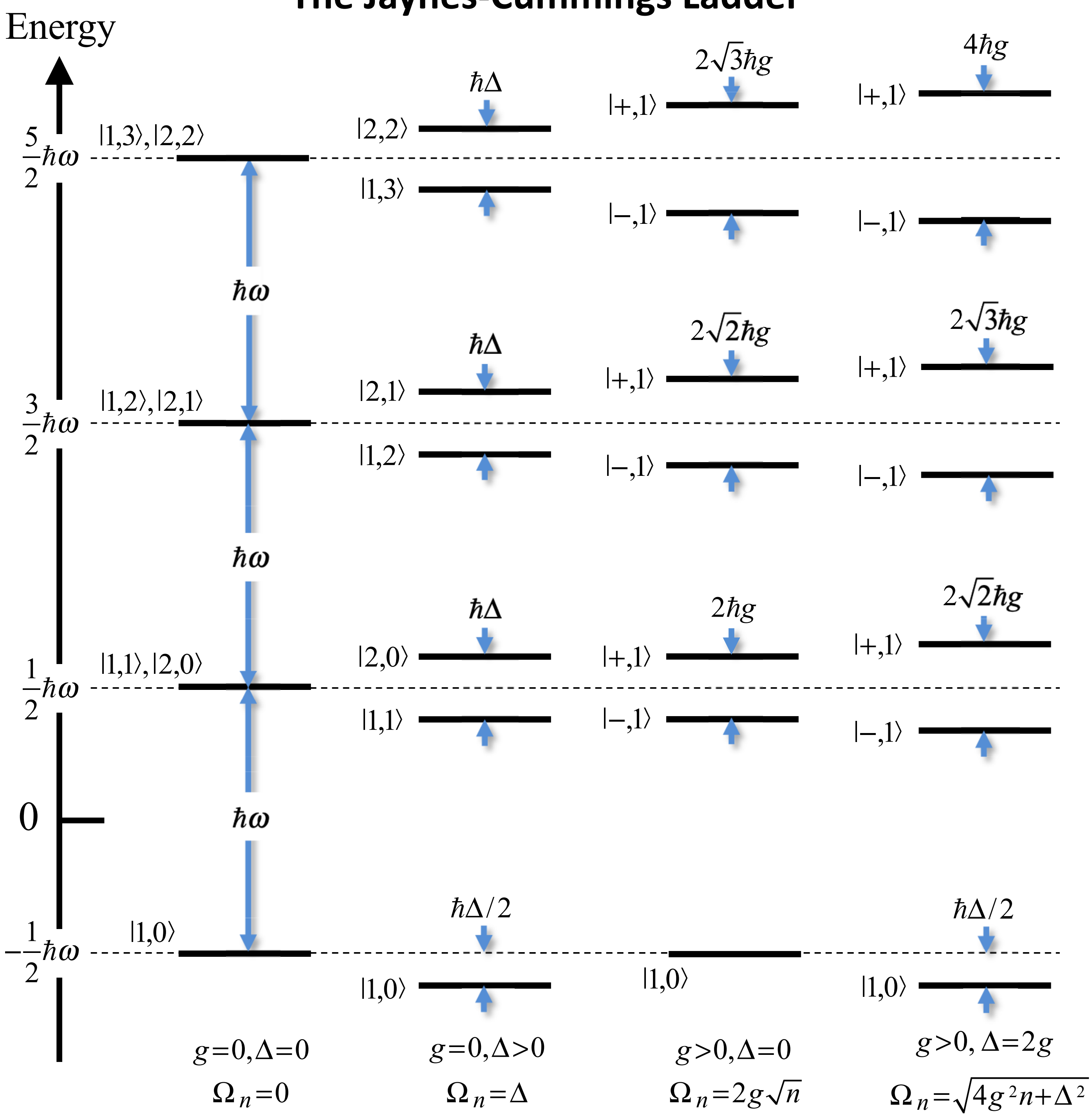
$$|-,n\rangle = -\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left(\theta_n/2\right)|1,n\rangle + \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(\theta_n/2\right)|2,n-1\rangle$$

for $\begin{smallmatrix}\Delta \le 0\\ \Delta > 0\end{smallmatrix}$

Mixing angle $\tan\theta_n = -\dfrac{2g\sqrt{n}}{\Delta}$

Energy Spectrum?

The Jaynes-Cummings Ladder

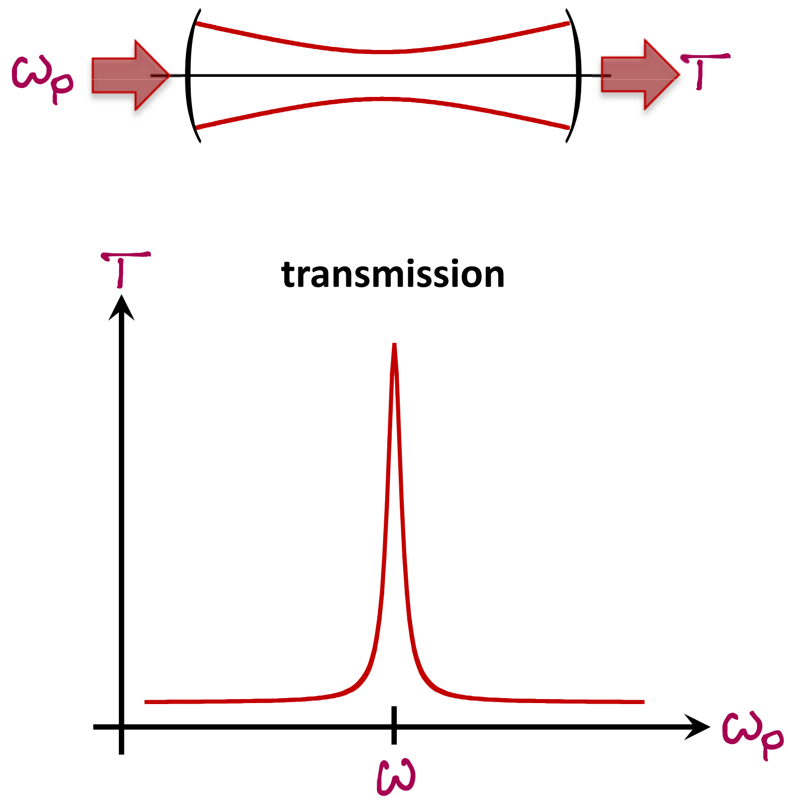


Phenomena Rooted in the Jaynes-Cummings Ladder

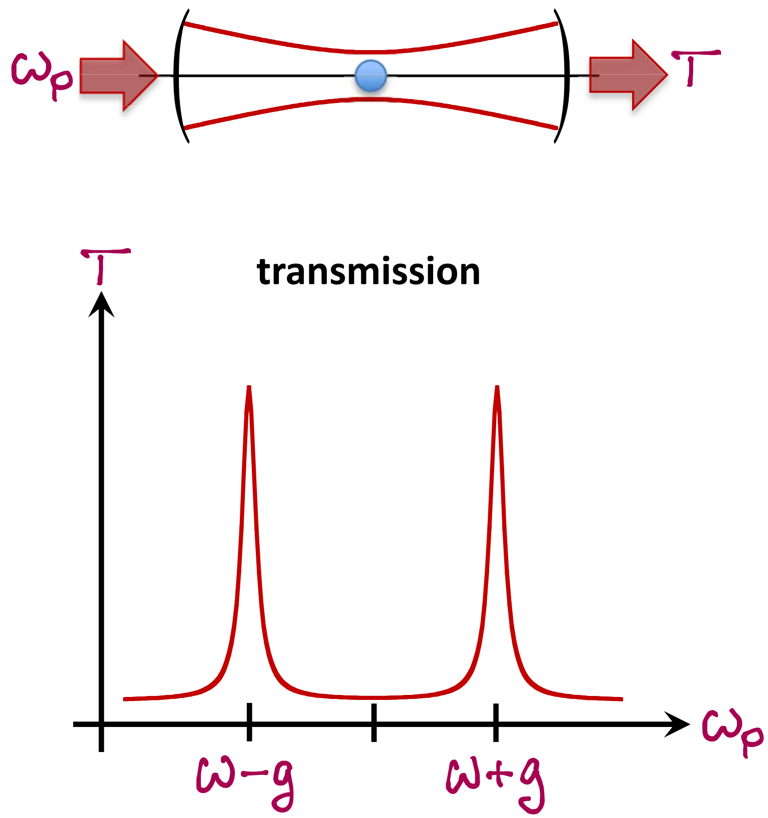
Vacuum Rabi splitting

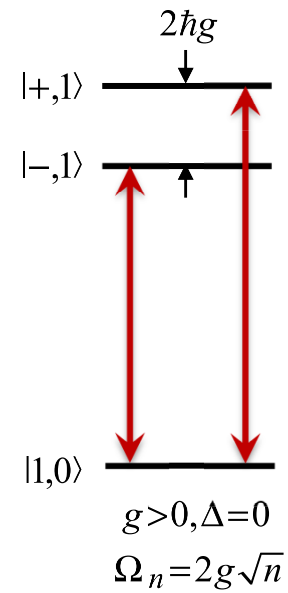
Consider the following experiments

empty Cavity



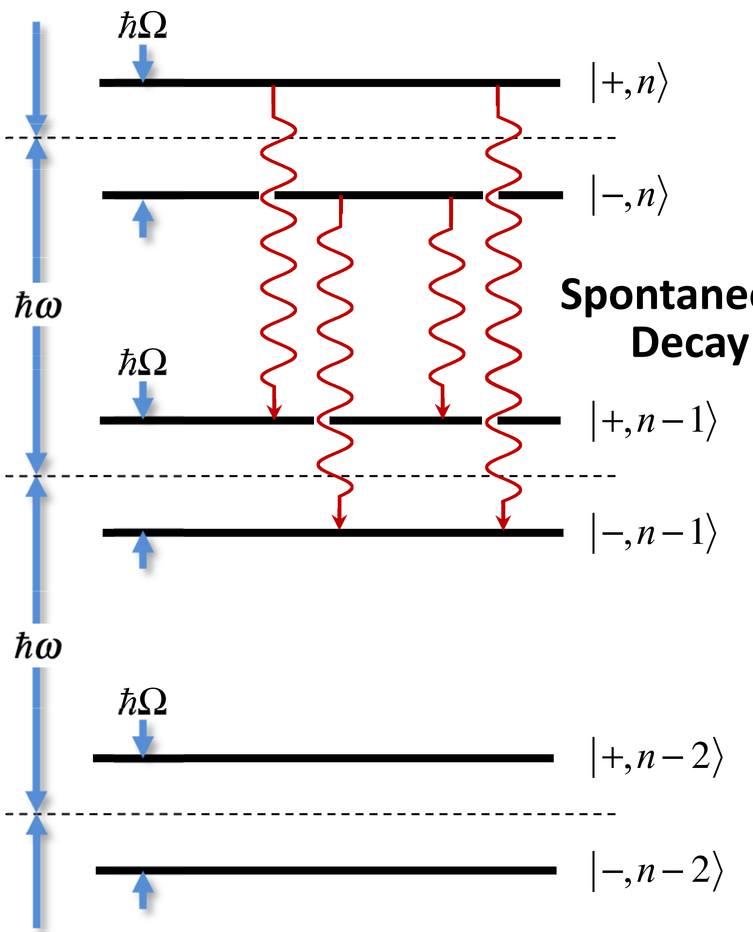
Cavity + Atom, Δ = 0,
one collective exciation





Fluorescence - Mollow Triplet

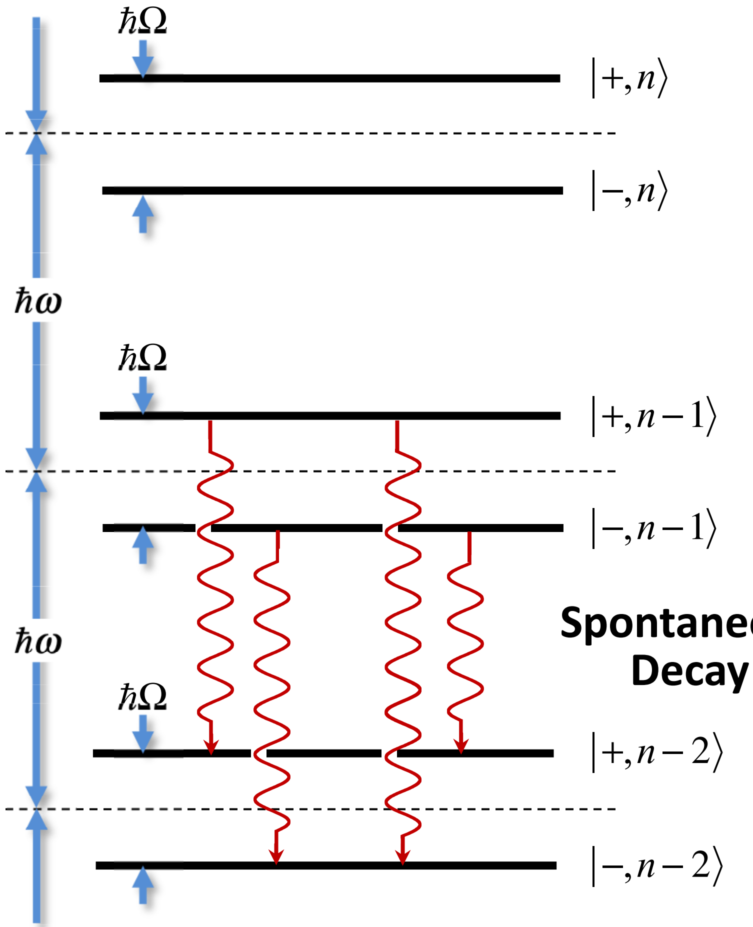
Coherent state with $\bar{n} \to \infty, \frac{\Delta n}{\bar{n}} \to 0, g \to 0$ ➡ $\Omega^2 = 4g^2(\bar{n} + \sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$



Spontaneous Decay ➡ **Fluorescence**

Fluorescence - Mollow Triplet

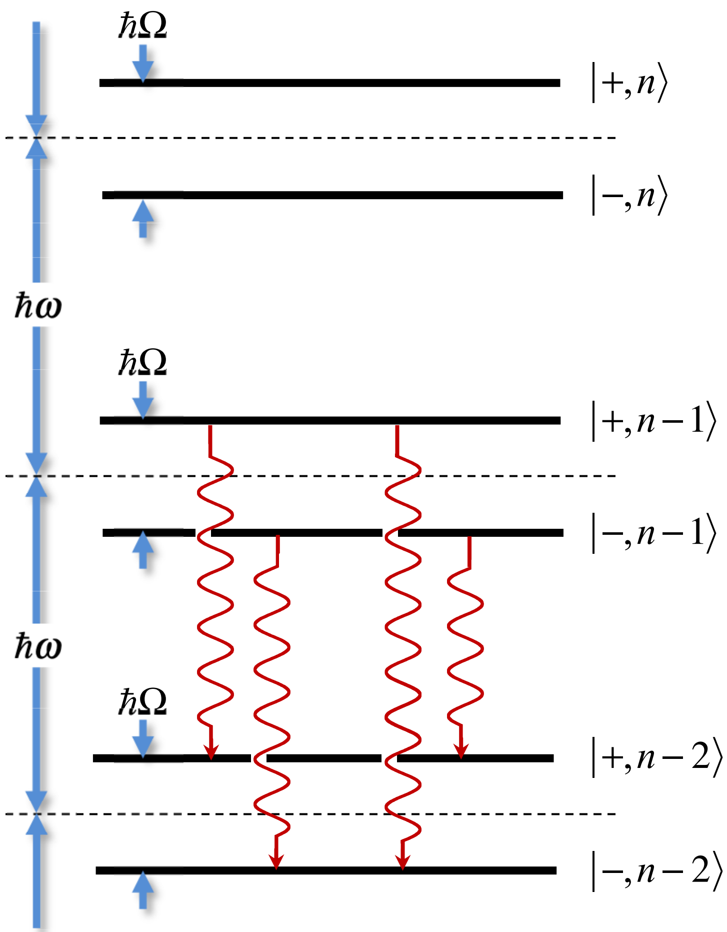
Coherent state with $\bar{n} \to \infty, \ \frac{\Delta n}{\bar{n}} \to 0, \ g \to 0 \ \Rightarrow \ \Omega^2 = 4g^2(\bar{n} + \sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$



Spontaneous Decay ⇒ **Fluorescence**

Fluorescence - Mollow Triplet

Coherent state with $\bar{n} = \infty, \frac{\Delta n}{\bar{n}} \to 0, g \to 0$ ➡ $\Omega^2 = 4g^2(\bar{n} + \sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$



Radiative Cascade

Mollow Triplet