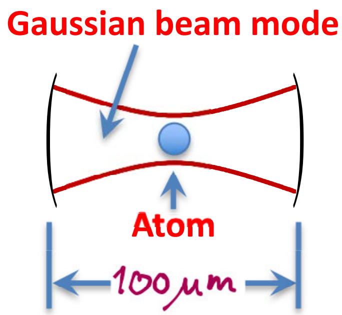
Quantized Light – Matter Interactions

More Cavity QED – Dressed States



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

Energy levels of the atom-cavity system

Bare & Dressed States

Return to single - mode result

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

$$\hbar\omega\hat{a}^{+}\hat{a} + \frac{1}{2}\omega_{21}\hat{\sigma}_z \quad \Rightarrow H_0$$

$$+\hbar g(\hat{\sigma}_+\hat{a}e^{i\Delta t} + \hat{\sigma}_-\hat{a}^{+}e^{-i\Delta t}) \quad \Rightarrow H_{AF}$$

"Bare" states ($g=0$, eigenstates of H_0)

State	Energy
$\|1,n\rangle$	$E_{1,n} = -\frac{\hbar\omega_{21}}{2} + n\hbar\omega$
$\|2,n-1\rangle$	$E_{2,n-1} = \frac{\hbar\omega_{21}}{2} + (n-1)\hbar\omega$

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Imagine we can tune ω_{21}

Energy level diagram

Crossings @ $\omega = \omega_{21}$ are degeneracies of pairs with n shared excitations

$n=0$ $\quad |1,0\rangle$

$n=1$ $\quad \{|1,1\rangle, |2,0\rangle\}$

$n=2$ $\quad \{|1,2\rangle, |2,1\rangle\}$

$\vdots$ $\quad \vdots$

n $\quad \{|1,n\rangle, |2,n-1\rangle\}$

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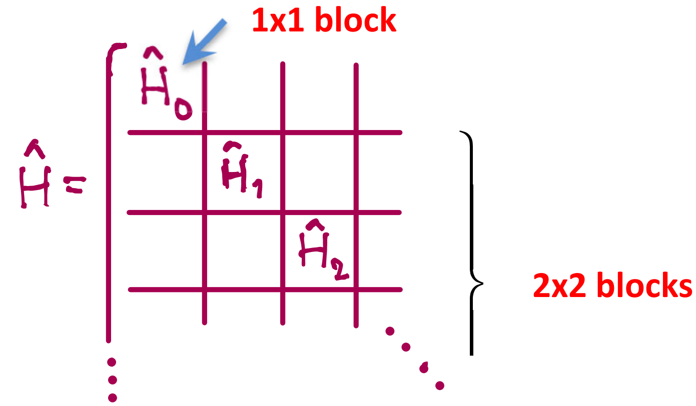
Energy level diagram

"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$:



Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n - \tfrac{1}{2})\hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

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"Dressed" states $\left\{ \begin{array}{l} \text{eigenstates of} \\ \hat{H} = \hat{H}_0 + \hat{H}_{AF} \end{array} \right.$

Structure of $\hat{H}$ **:**

1x1 block

$$\hat{H} = \begin{bmatrix} \hat{H}_0 & & & \\ & \hat{H}_1 & & \\ & & \hat{H}_2 & \\ \vdots & & & \ddots \end{bmatrix}$$

2x2 blocks

Can write this on the form

$$\boxed{\hat{H}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (n - \tfrac{1}{2}) \hbar\omega + \begin{bmatrix} -\hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & \hbar\Delta/2 \end{bmatrix}}$$

Eigenvalues $E_\pm = (n - \tfrac{1}{2})\hbar\omega \pm \frac{\hbar}{2}\sqrt{4g^2 n + \Delta^2}$

Eigenstates

$$|+,n\rangle = \substack{\cos \\ \sin}\left(\theta_n/2\right)|1,n\rangle + \substack{\sin \\ \cos}\left(\theta_n/2\right)|2,n-1\rangle$$

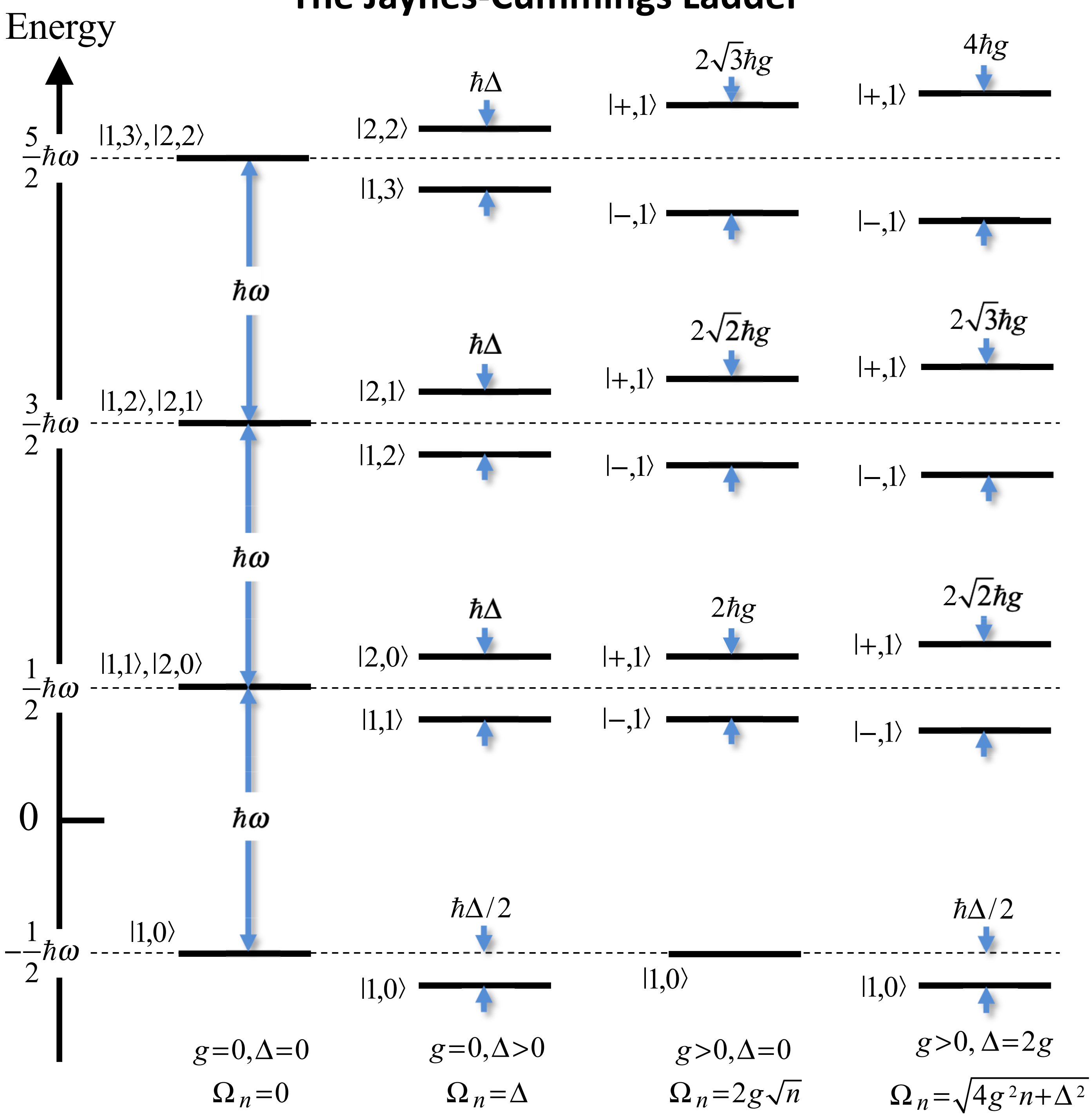
$$|-,n\rangle = -\substack{\sin \\ \cos}\left(\theta_n/2\right)|1,n\rangle + \substack{\cos \\ \sin}\left(\theta_n/2\right)|2,n-1\rangle$$

for $\substack{\Delta \leq 0 \\ \Delta > 0}$

Mixing angle $\tan\theta_n = -\dfrac{2g\sqrt{n}}{\Delta}$

Energy Spectrum?

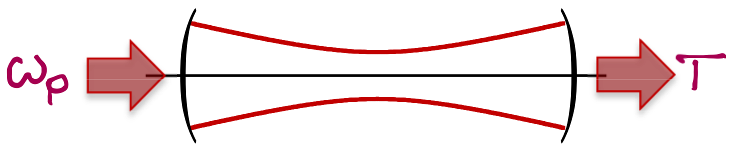
The Jaynes-Cummings Ladder



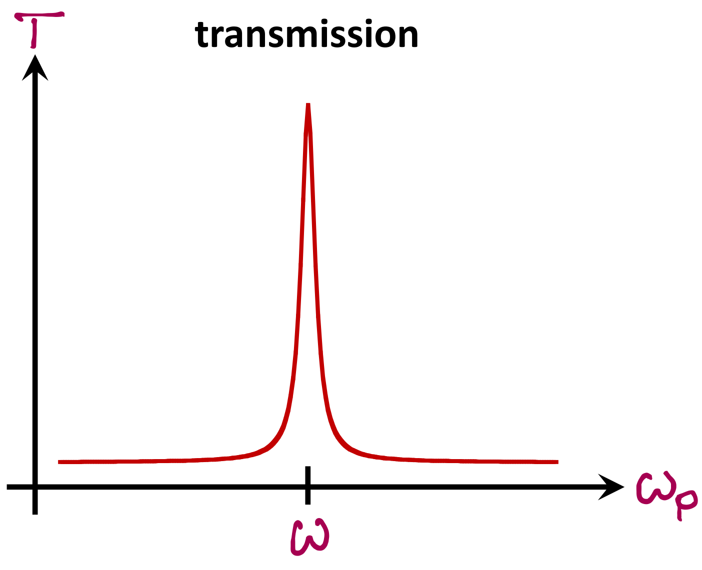
Vacuum Rabi splitting

Consider the following experiments

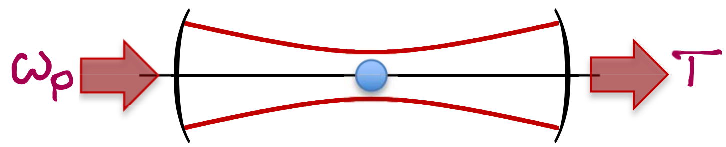
empty Cavity



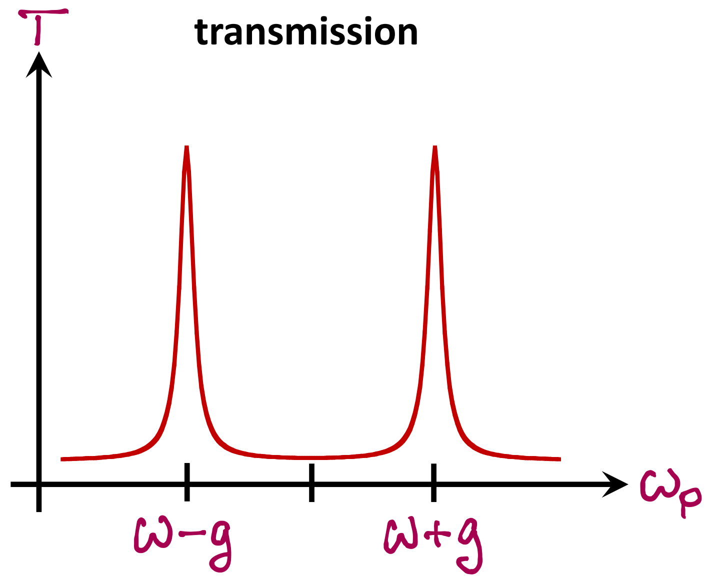
transmission

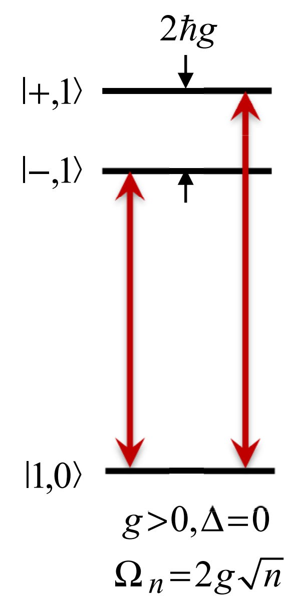


Cavity + Atom, $\Delta = 0$,
one collective exciation



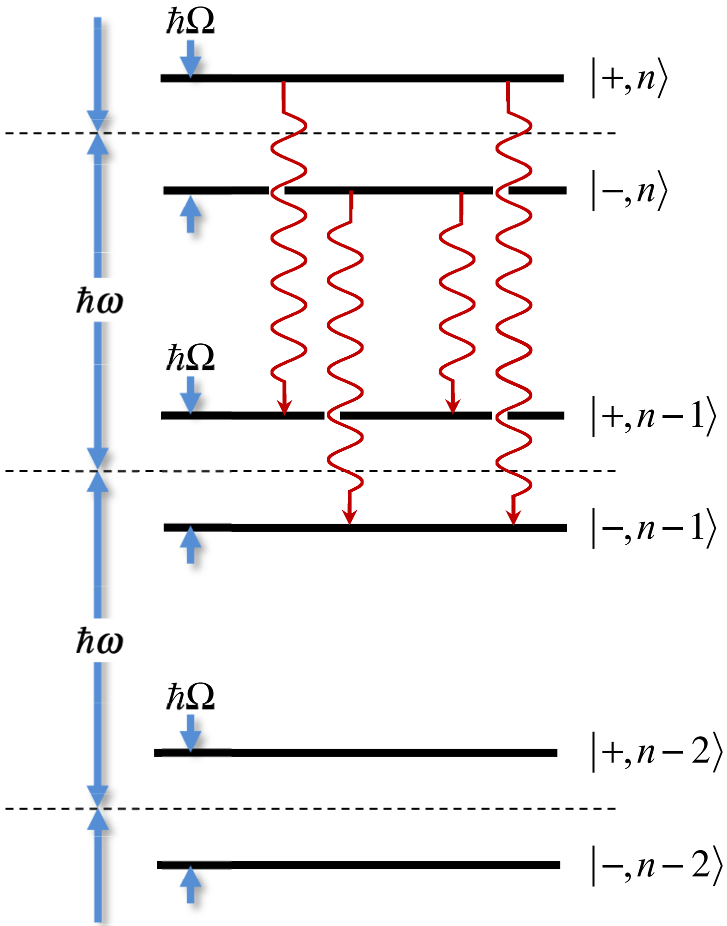
transmission



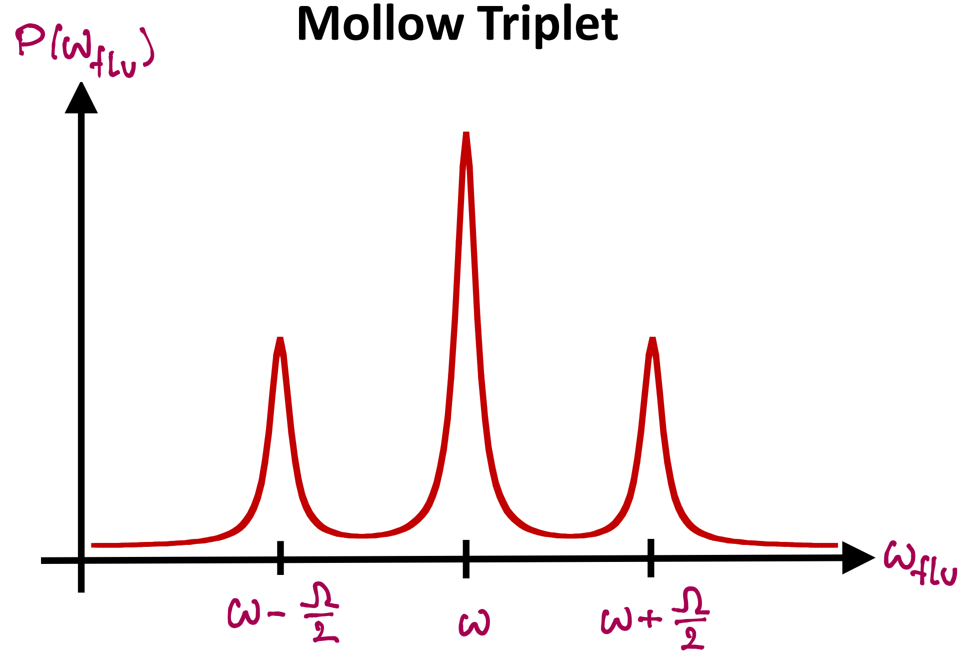


Jaynes Cummings Ladder

Coherent state with $\bar{n} = \infty,\ \frac{\Delta n}{\bar{n}} \to 0,\ g \to 0 \Rightarrow \Omega^2 = 4g^2(\bar{n} + \sqrt{\bar{n}}) + \Delta^2 \sim 4g^2\bar{n} + \Delta^2$



Mollow Triplet



Wigner-Weisskopf Theory of Spontaneous Decay

Today: Decay of atomic excited state due to interaction with the quantum electromagnetic field

Setup

Hamiltonian: (Schrödinger Picture)

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} =$$

$$\sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}} \hbar\left(g_{\vec{k}}\hat{\sigma}_+ \hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_- \hat{a}^{\dagger}_{\vec{k}}\right)$$

Hamiltonian: (Interaction Pict., Res. Approx.)

$$\hat{H}_I(t) = \sum_{\vec{k},\lambda} \hbar g_{\vec{k},\lambda} \hat{\sigma}_+ \hat{a}_{\vec{k},\lambda} e^{i(\omega_{21}-\omega_{\vec{k}})t} + H.c.$$

S. E.: $i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}_I(t) |\psi_I(t)\rangle$

Expand

$$|\psi(t)\rangle = C_{2,0}(t)|2,0\rangle + \sum_{\vec{k},\lambda} C_{1,1_{\vec{k},\lambda}}(t)|1,1_{\vec{k},\lambda}\rangle$$

$$\dot{C}_{2,0}(t) = -i \sum_{\vec{k},\lambda} g_{\vec{k},\lambda} e^{i(\omega_{21}-\omega_{\vec{k}})t} C_{1,1_{\vec{k},\lambda}}(t)$$

$$\dot{C}_{1,1_{\vec{k},\lambda}}(t) = -i\, g^*_{\vec{k},\lambda} e^{-i(\omega_{21}-\omega_{\vec{k}})t} C_{2,0}(t)$$

infinite # of these

Formal Solution:

$$C_{1,1_{\vec{k},\lambda}}(t) = -i\, g^*_{\vec{k},\lambda} \int_0^t e^{-i(\omega_{21}-\omega_{\vec{k}})t'} C_{2,0}(t')\,dt'$$

Wigner-Weisskopf Theory of Spontaneous Decay

Expand

$$|\psi(t)\rangle = c_{2,0}(t)|2,0\rangle + \sum_{\vec{k},\lambda} c_{1,1_{\vec{k},\lambda}}(t)|1,1_{\vec{k},\lambda}\rangle$$

$$\dot{c}_{2,0}(t) = -i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda} e^{i(\omega_{21}-\omega_{\vec{k}})t} c_{1,1_{\vec{k},\lambda}}(t)$$

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$$\dot{c}_{2,0}(t) = -\sum_{\vec{k},\lambda}|g_{\vec{k},\lambda}|^2\int_0^t e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt'$$

Time Dep. Perturbation Theory:

Study short-time limit, $c_{2,0}(t)\sim 1$

⇨ Fermi's Golden Rule

No good for this problem!

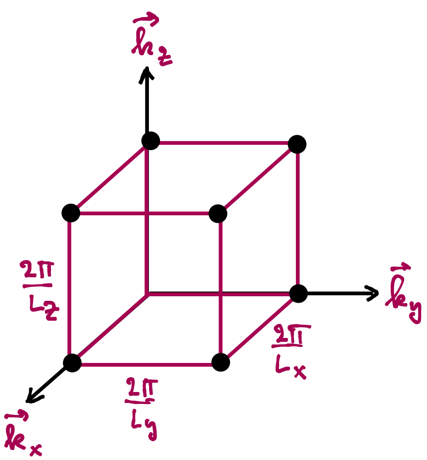
Wigner-Weisskopf Theory of Spontaneous Decay

$$\boxed{\dot{c}_{2,0}(t) = -\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \int_0^t e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')}\, c_{2,0}(t')\,dt'}$$

Quantization in a box w/periodic B. C.

$\Rightarrow$ $k_i = n_i \frac{2\pi}{L}$, n_i **integer**

In $\vec{k}$ space the modes form a grid:



$$\frac{1 \text{ mode}}{\left(\frac{2\pi}{L_x}\right)\left(\frac{2\pi}{L_y}\right)\left(\frac{2\pi}{L_z}\right)}$$

$$= \frac{V}{(2\pi)^3} = \mathcal{D}(\vec{k})$$

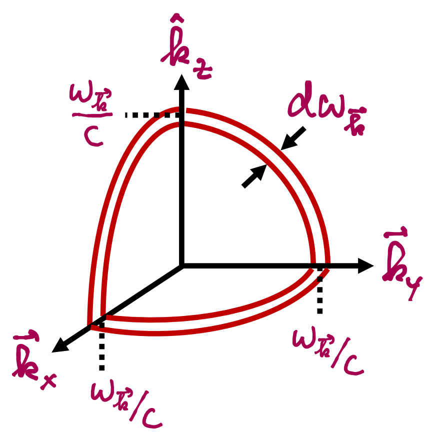
Density of Modes

Convert sum to integral over modes:

$$\sum_{\vec{k}} \rightarrow \int d^3\vec{k}\, \mathcal{D}(\vec{k}) = \int k^2 d(\hat{k})\, dk\, \mathcal{D}(\vec{k})$$

$$= \int d(\hat{k})\, d\omega_{\vec{k}} \frac{\omega_{\vec{k}}^2}{c^3} \mathcal{D}(\vec{k}) = \int d(\hat{k})\, d\omega_{\vec{k}}\, \mathcal{D}(\omega_{\vec{k}})$$

where $\hat{k} = \vec{k}/k$ is a unit vector along $\vec{k}$ and



$$\boxed{\mathcal{D}(\omega_{\vec{k}}) = \frac{V}{(2\pi)^3} \frac{\omega_{\vec{k}}^2}{c^3}}$$

mode density in shell of $\vec{k}$ - space of radius $\omega_{\vec{k}}/c$

Wigner-Weisskopf Theory of Spontaneous Decay

$$\dot{c}_{2,0}(t) = -\sum_{\vec k,\lambda} |g_{\vec k,\lambda}|^2 \int_0^t e^{i(\omega_{21}-\omega_{\vec k})(t-t')} c_{2,0}(t')\,dt'$$

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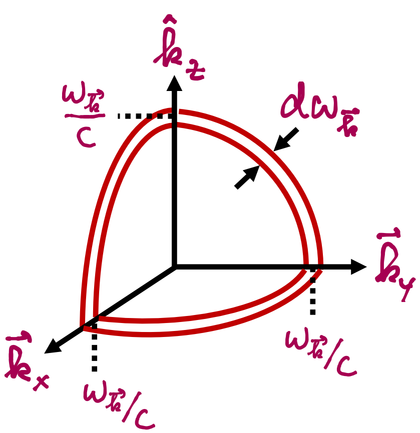
where $\hat k = \vec k / k$ **is a unit vector along** $\vec k$ **and**



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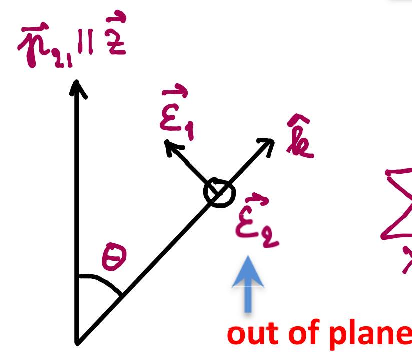
mode density in shell of $\vec k$ **- space of radius** $\omega_{\vec k}/c$

Thus, in the Continuum Limit

$$\sum_{\vec k,\lambda} |g_{\vec k,\lambda}|^2 \to \int_0^\infty d\omega_{\vec k}\, \mathcal{D}(\omega_{\vec k}) \sum_\lambda \int d(\hat k)\, |g_{\vec k,\lambda}|^2$$

We define the "polarization average"

$$\overline{|g(\omega_{\vec k})|^2} = \sum_\lambda \int d(\hat k)\, |g_{\vec k,\lambda}|^2 = \frac{1}{\hbar^2}\underbrace{\left(\frac{\hbar\omega_{\vec k}}{2\varepsilon_0 V}\right)}_{\mathcal{E}_{\vec k}^2}\int d(\hat k) \sum_\lambda |\vec p_{21}\cdot\vec\varepsilon_{\vec k,\lambda}|^2$$



in polar coordinates

$$\sum_\lambda |\vec p_{21}\cdot\vec\varepsilon_{\vec k,\lambda}|^2 = \sin^2\theta\, |\vec p_{21}|^2$$

no φ dependence

$$\int d(\hat k) \sum_\lambda |\vec p_{21}\cdot\vec\varepsilon_{\vec k,\lambda}|^2 = \int_0^{2\pi} d\phi \int_0^1 d(\cos\theta)\sin^2\theta\, |\vec p_{21}|^2$$

$$= 2\pi |\vec p_{21}|^2 \int_{-1}^1 du\,(1-u^2) = \frac{8\pi}{3}|\vec p_{21}|^2$$

Wigner-Weisskopf Theory of Spontaneous Decay

$$\boxed{\dot{c}_{2,0}(t) = -\boxed{\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2} \int_0^t e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt'}$$

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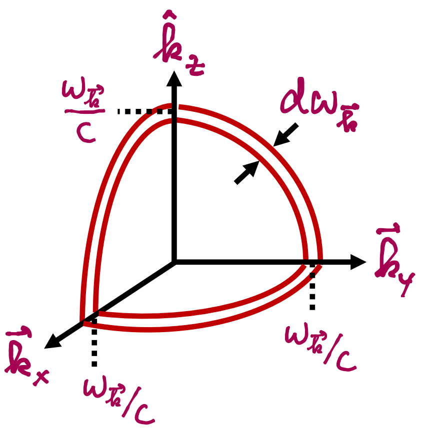

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$$\frac{1}{\hbar^2}\left(\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}\right) \underbrace{\int d(\hat{k}) \sum_\lambda |\vec{\wp}_{21}\cdot\vec{\varepsilon}_{\vec{k},\lambda}|^2}$$

$$= \frac{4\pi\omega_{\vec{k}}}{2\hbar\varepsilon_0 V} |\vec{\wp}_{21}|^2$$

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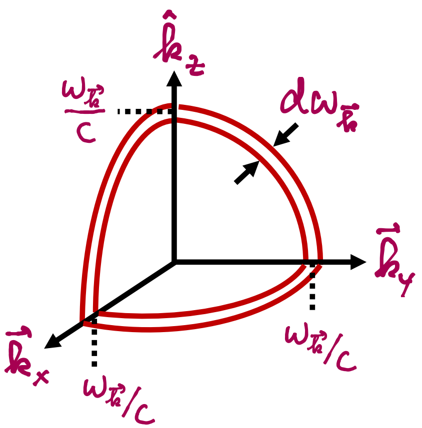

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$$= \frac{4\pi\omega_{\vec{k}}}{2\hbar\varepsilon_0 V}|\vec{p}_{21}|^2$$

Putting it together:

$$\dot{c}_{2,0}(t) = -\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \int_0^t e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt'$$

$$= -\int_0^\infty d\omega_{\vec{k}}\,\overline{|g(\omega_{\vec{k}})|^2}\,\mathcal{D}(\omega_{\vec{k}})\int_0^t dt'\, e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')$$

Wigner-Weisskopf Theory of Spontaneous Decay

Thus, in the Continuum Limit

$$\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \to \int_0^\infty d\omega_{\vec{k}}\, \mathcal{D}(\omega_{\vec{k}}) \sum_\lambda \int d(\hat{k})\, |g_{\vec{k},\lambda}|^2$$

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Putting it together:

$$\boxed{\begin{aligned}\dot{c}_{2,0}(t) &= -\sum_{\vec{k},\lambda} |g_{\vec{k},\lambda}|^2 \int_0^t e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\,dt' \\ &= -\int_0^\infty d\omega_{\vec{k}}\, \overline{|g(\omega_{\vec{k}})|^2}\, \mathcal{D}(\omega_{\vec{k}}) \int_0^t dt'\, e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t')\end{aligned}}$$

Dr. Wigner & Dr. Weisskopf:

Atoms couple weakly to the vacuum

$c_{2,0}(t)$ changes slowly on timescale ω_{21}^{-1}, evolving at a rate Γ to be determined.

In eq. for $\dot{c}_{2,0}$ we take $c_{2,0}(t') \sim$ constant on timescales $\omega_{21}^{-1} \ll t \ll \Gamma^{-1}$

Wigner-Weisskopf approximation

$$\boxed{\dot{c}_{2,0}(t) \approx -\int_0^\infty d\omega_{\vec{k}}\, \overline{|g(\omega_{\vec{k}})|^2}\, \mathcal{D}(\omega_{\vec{k}}) \times \int_0^t dt'\, e^{i(\omega_{21}-\omega_{\vec{k}})(t-t')} c_{2,0}(t)}$$

To be Continued…