Quantized Light – Matter Interactions

General Problem:

Quantum Atom $\longleftrightarrow$ Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^{+}_{\vec{k}} \hat{a}_{\vec{k}} + 1/2\right) \qquad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}} \cdot \hat{\vec{E}}(\vec{r},t) \qquad \text{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij} \qquad (2)$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right) \qquad (3)$$

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Dipole Operator:

(2) $$\boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\hat{\sigma}_{ij}}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

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(3) $$\boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij}\sum_{\vec{k}} -\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}\hat{\sigma}_{ij}(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

$$= \sum_{ij}\sum_{\vec{k}} \hbar g^{(ij)}_{\vec{k}} (\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})$$

where $$g^{(ij)}_{\vec{k}} = \frac{\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}}{\hbar}$$

Rabi Freq., note sign convention

2-level atom ⇨ $(i,j) = (1,2)$:

$$\boxed{\hat{H}_{AF} = \hbar\sum_{\vec{k}} (g_{\vec{k}}\hat{\sigma}_{21} + g^{*}_{\vec{k}}\hat{\sigma}_{12})(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}})}$$

Define:

$$\hat{\sigma}_+ = \hat{\sigma}_{21} = |2\rangle\langle 1|$$
$$\hat{\sigma}_- = \hat{\sigma}_{12} = |1\rangle\langle 2|$$
$$\hat{\sigma}_z = \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1|$$

⇨ Pauli matrices

$$\hat{\sigma}_x = \frac{1}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)$$
$$\hat{\sigma}_y = \frac{1}{2i}(\hat{\sigma}_+ - \hat{\sigma}_-)$$
$$\hat{\sigma}_z$$

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Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij}\sum_{\vec{k}} -\vec{\eta}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}\hat{\sigma}_{ij}(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}})$$

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Pauli matrices

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$\hat{\sigma}_z$

With this notation **(4)**

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}(g_{\vec{k}}\hat{\sigma}_+\hat{a}_{\vec{k}} + \cancel{g_{\vec{k}}\hat{\sigma}_+\hat{a}^+_{\vec{k}}} + \cancel{g^*_{\vec{k}}\hat{\sigma}_-\hat{a}_{\vec{k}}} + g^*_{\vec{k}}\hat{\sigma}_-\hat{a}^+_{\vec{k}})$$

Energy conservation?

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}(g_{\vec{k}}\hat{\sigma}_+\hat{a}_{\vec{k}} + g^*_{\vec{k}}\hat{\sigma}_-\hat{a}^+_{\vec{k}})$$

Putting it all together

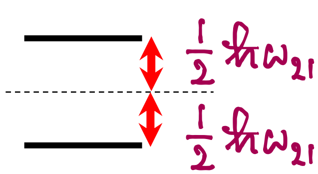
$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \qquad (5)$$

$$\sum_{\vec{k}}\hbar\omega_{\vec{k}}\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z + \sum_{\vec{k}}\hbar(g_{\vec{k}}\hat{\sigma}_+\hat{a}_{\vec{k}} + g^*_{\vec{k}}\hat{\sigma}_-\hat{a}^+_{\vec{k}})$$

We changed the zero point for energy by subtracting

$\sum_{\vec{k}}\frac{1}{2}\hbar\omega_{\vec{k}}$ and $\frac{1}{2}(E_2 - E_1)$

field atom



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With this notation (4)

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}^+_{\vec{k}}} + \cancel{g^*_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g^*_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right)$$

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Putting it all together

$$\boxed{\begin{aligned} &\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} = \qquad (5) \\ &\sum_{\vec{k}} \hbar \omega_{\vec{k}} \hat{a}^+_{\vec{k}} \hat{a}^+_{\vec{k}} + \tfrac{1}{2} \hbar \omega_{21} \hat{\sigma}_z + \sum_{\vec{k}} \hbar \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + g^*_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right) \end{aligned}}$$

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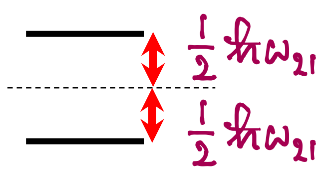

Foundational result for remainder of course

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With this notation

$$\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}} \hat{\sigma}_+ \hat{a}_{\vec{k}} + \cancel{g_{\vec{k}} \hat{\sigma}_+ \hat{a}^+_{\vec{k}}} + \cancel{g^*_{\vec{k}} \hat{\sigma}_- \hat{a}_{\vec{k}}} + g^*_{\vec{k}} \hat{\sigma}_- \hat{a}^+_{\vec{k}} \right) \qquad (4)$$

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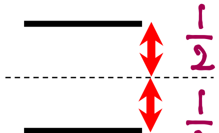

$\frac{1}{2}\hbar\omega_{21}$

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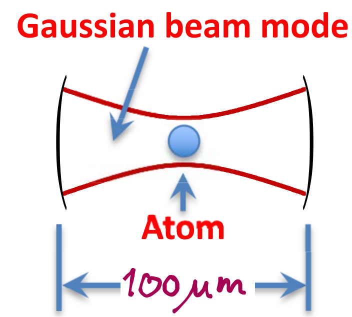
Interaction with Single-mode Fields

Good approx. in small, high-Q Cavity



$c/2L \gg A_{21}$

$|g_{\vec{k}}| \gg A_{21}, \gamma$

Single-mode (Jaynes-Cummings) Hamiltonian

$$\boxed{\hat{H} = \underbrace{\hbar\omega \hat{a}^+\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar\left(g_{\vec{k}}\hat{\sigma}_+ + g^*_{\vec{k}}\hat{\sigma}_-\right)\left(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}}\right)}_{H_{AF}}}$$

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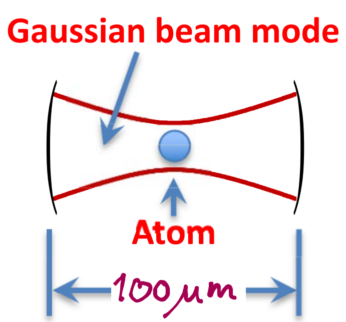

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Paradigm for spin-1/2 coupled to QHO

- Atom in high-Q cavity *)
- Quantum dot in high-Q
- Rydberg atom in superconducting μw Cavity
- Superconducting qubit in superconducting μw cavity
- Superconducting qubit in superconducting μw stripline cavity (circuit QED)
- Trapped ion with quantized COM motion *)

*) Nobel Prize in Physics 2012

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More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\,\hat{a}^{+}\hat{a} + \tfrac{1}{2}\hbar\omega_{21}\hat{\sigma}_{z}}_{H_0} + \underbrace{\hbar\left(g_{\vec{k}}\hat{\sigma}_{+} + g_{\vec{k}}^{*}\hat{\sigma}_{-}\right)\left(\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^{+}\right)}_{H_{AF}}$$

For simplicity $\vec{r}_{21} = \vec{r}_{12} \Rightarrow g_{\vec{k}} = g_{\vec{k}}^{*} = g_{\vec{k}}$

Note: $\hat{H}_{AF}$ conserves excitation number, couples $|2,n\rangle \leftrightarrow |1,n+1\rangle$

Series of 2-level systems, one for each n

All 2-level systems are alike
Rabi problem!

Switch to Interaction Picture, 6-7 in Notes:

$$\left.\begin{aligned} \hat{H}_S &\rightarrow \hat{H}_I = e^{i\frac{\hat{H}_0}{\hbar}t}\,\hat{H}_{AF}\,e^{-i\frac{\hat{H}_0}{\hbar}t} \\ |\psi_S(t)\rangle &\rightarrow |\psi_I(t)\rangle = e^{i\frac{\hat{H}_0}{\hbar}t}\,|\psi_S(t)\rangle \end{aligned}\right\} \Rightarrow$$

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More about Single-mode Cavity QED

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^+\hat{a} + \frac{1}{2}\hbar\omega_{21}\hat{\sigma}_z}_{H_0} + \underbrace{\hbar(g_{\vec{k}}\hat{\sigma}_+ + g_{\vec{k}}^*\hat{\sigma}_-)(\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}^+)}_{H_{AF}}$$

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⇓

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$$\hat{H}_{AF} = \hbar g(\hat{\sigma}_+\hat{a}e^{i(\omega_{21}-\omega)t} + \hat{\sigma}_+\hat{a}^+e^{i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}e^{-i(\omega_{21}+\omega)t} + \hat{\sigma}_-\hat{a}^+e^{-i(\omega_{21}-\omega)t})$$

RWA and resonant approximation

⇓

Jaynes-Cummings Hamiltonian

$$\boxed{\hat{H}_I = \hat{H}_{AF} = \hbar g(\hat{\sigma}_+\hat{a}e^{i\Delta t} + \hat{\sigma}_-\hat{a}^+e^{-i\Delta t})}$$

$\Delta = \omega_{21} - \omega$

End 04-23-2021

Can show

$$e^{i\omega\hat{a}^+\hat{a}t}\,\hat{a}\,e^{-i\omega\hat{a}^+\hat{a}t} = \hat{a}e^{-i\omega t}$$

$$e^{i\frac{\omega_{21}}{2}\hat{\sigma}_z t}\,\hat{\sigma}_+\,e^{-i\frac{\omega_{21}}{2}\hat{\sigma}_z t} = \hat{\sigma}_+e^{i\omega_{21}t}$$