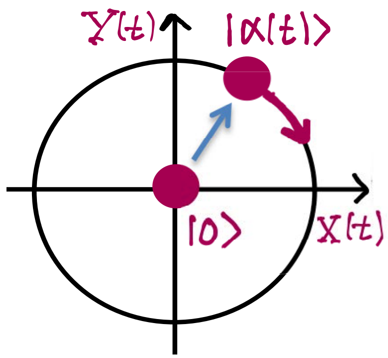
Quantum States of the Quantized Field

More about Coherent States



Coherent States as translated Vacuum States?

Generating Coherent States from the Vacuum

<u>**Definition**</u>: $\hat{D}(\alpha) = e^{\alpha \hat{a}^{+} - \alpha^{*} \hat{a}}$

Unitary, equals translation

Glaubers formula (from BCH formula)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Apply to $[\underbrace{\alpha \hat{a}^{+}}_{\hat{A}}, \underbrace{-\alpha^{*} \hat{a}}_{\hat{B}}] = \underbrace{\alpha^{*}\alpha}_{[\hat{A},\hat{B}]}$

$$\boxed{\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^{+}} e^{-\alpha \hat{a}}}$$

Remember: $\hat{a}|0\rangle = 0$

$$e^{-\alpha^{*}\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^{*}\hat{a})^n}{n!}|0\rangle = |0\rangle$$

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^{+}}|0\rangle$$
$$= e^{-|\alpha|^2} \sum_n \frac{(\alpha \hat{a}^{+})^n}{n!}|0\rangle$$
$$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

$$\boxed{\hat{D}(\alpha)|0\rangle = |\alpha\rangle}$$

Quantum States of the Quantized Field

Apply to $[\alpha\hat{a}^{+}, -\alpha^{*}\hat{a}] = \alpha^{*}\alpha$

$\hat{A}$ $\quad \hat{B}$ $\quad [\hat{A},\hat{B}]$

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$$\boxed{\hat{D}(\alpha)|0\rangle = |\alpha\rangle}$$

OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat{a}^{+} - \alpha^{*}\hat{a} = (\alpha-\alpha^{*})\hat{X} + i(\alpha+\alpha^{*})\hat{Y}$$

$$= i2Y\hat{X} + i2X\hat{Y}$$

where $X = \langle\alpha|\hat{X}|\alpha\rangle$, $Y = \langle\alpha|\hat{Y}|\alpha\rangle$

Glaubers formula again:

$$\boxed{\hat{D}(\alpha) = e^{i2Y\hat{X} + i2X\hat{Y}} = e^{-XY/4} e^{i2Y\hat{X}} e^{i2X\hat{Y}}}$$

Recall: $\hat{S}(q) = e^{-iq\hat{p}/\hbar}$ translation by q

$\hat{S}(p) = e^{-ip\hat{q}/\hbar}$ translation by p

where $q = q_0 X$, $p = p_0 Y$ & $x_0 p_0 = 2\hbar$

$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$

Quantum States of the Quantized Field

OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

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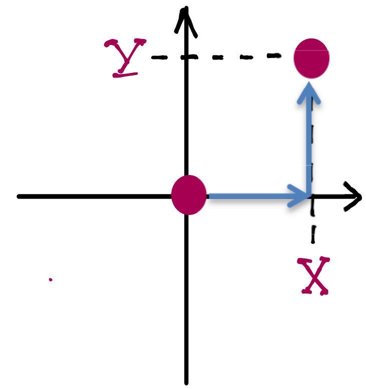
$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$

This gives us

$$\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}$$

$\hat{D}(\alpha)$ translates along X then Y



Discussion –
How to do this?

Quantum States of the Quantized Field

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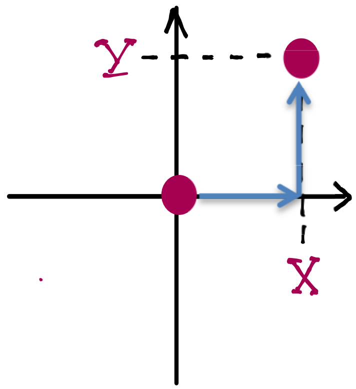

Discussion – How to do this?

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t)$ @ $t = 0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k (\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\boxed{\begin{gathered}\hat{H} = \hbar\omega_k(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a} + \hat{a}^+) \\ \lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)\end{gathered}}$$

Homework Problem (voluntary)

$$\boxed{\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega_k)T/2} \frac{\sin[(\omega-\omega_k)T/2]}{(\omega-\omega_k)/2}}$$

Quantum States of the Quantized Field

Coherent States from Classical Dipole Radiation

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Dipole-Field Interaction

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a} + \hat{a}^+)$$

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Homework Problem (voluntary)

$$\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega_k)T/2} \frac{\sin[(\omega-\omega_k)T/2]}{(\omega-\omega_k)/2}$$

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ drives $\hat{E}(t)$

classical dipole + quantum fluctuations

coherent state + quantum fluctuations

Drive from $0 < t < T$

$$\alpha(t) = \alpha(T) e^{-i\omega_k(t-T)}$$

Quantum States of the Quantized Field

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ drives $\hat{E}(t)$

classical dipole
+ quantum fluctuations

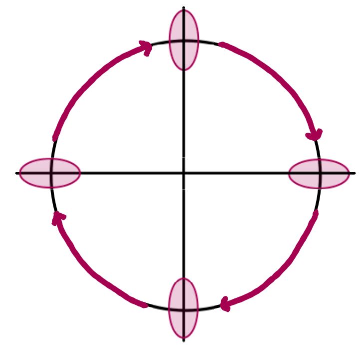
coherent state
+ quantum fluctuations

Squeezed States

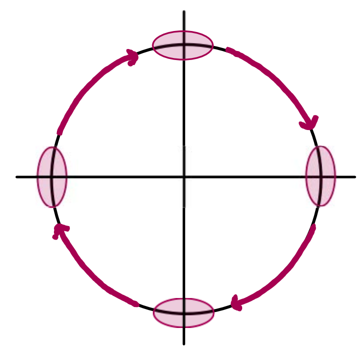
Minimum uncertainty states w/asymmetry

$\Delta X \Delta Y = 1/4, \quad \Delta X(t) \neq \Delta Y(t)$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium

Quantum States of the Quantized Field

Odds and Ends – Thermal States

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_B T}\right]$$

$$\hat{\rho} = \sum_n P(n)|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-E_n/k_B T}|n\rangle\langle n|$$

$$= (1-q)\sum_n q^n |n\rangle\langle n|, \quad q = e^{-\hbar\omega/k_B T}$$

Mean Photon Number:

$$\bar{n} = \mathrm{Tr}(\hat{\rho}\hat{N}) = \sum_{k,n}\langle k|(1-q)q^n|n\rangle\langle n|\hat{N}|k\rangle$$

$$= (1-q)\sum_n n q^n = \frac{q}{1-q}$$

Photon Number Uncertainty:

$$\langle \hat{N}^2\rangle = (1-q)\sum_n n^2 q^n = \frac{q^2+q}{(1-q)}$$

$$\Delta n^2 = \langle \hat{N}^2\rangle - \langle \hat{N}\rangle^2$$

$$= \frac{q^2+q}{(1-q)} - \frac{q^2}{(1-q)^2} = \frac{q}{(1-q)^2}$$

$$\bar{n} = \frac{q}{1-q}$$

Coherent State limit

$$\Delta n = \frac{\sqrt{q}}{1-q} = \sqrt{\bar{n}(\bar{n}+1)} \geq \sqrt{\bar{n}}$$

Optical Frequencies, Room Temperature:

$$\lambda = 1\,\mu m, \quad T = 300\,K$$

$$q = 6.5\times 10^{-6}, \quad \bar{n} \sim 10^{-6}$$

Quantum States of the Quantized Field

Odds and Ends – Quantum-Classical Correspondence

Define a Translation Operator

$$\hat{T}_\alpha(t) = e^{\alpha^* e^{i\omega t}\hat{a} - \alpha e^{-i\omega t}\hat{a}^+} = \hat{D}(-\alpha e^{-i\omega t})$$

Use $[\hat{a}, \hat{F}(\hat{a}^+)] = dF(\hat{a}^+)/d\hat{a}^+$ **to show**

$$[\hat{a}, \hat{T}_\alpha] = \hat{a}\hat{T}_\alpha - \hat{T}_\alpha\hat{a} = -\alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha\hat{a} = \hat{a}\hat{T}_\alpha + \alpha e^{-i\omega t}\hat{T}_\alpha$$

$$\Rightarrow \hat{T}_\alpha\hat{a}\hat{T}_\alpha^+ = \hat{a} + \alpha e^{-i\omega t}$$

From this we get

$$\hat{E}_\perp' = \hat{T}_\alpha\hat{E}_\perp\hat{T}_\alpha^+ = \hat{T}_\alpha\left(\mathcal{E}_k\hat{a}e^{i\vec{k}\cdot\vec{r}} + H.C.\right)\hat{T}_\alpha^+$$

$$= \mathcal{E}_k\hat{a}e^{i\vec{k}\cdot\vec{r}} + H.C. + \mathcal{E}_k\alpha e^{-i(\omega t - \vec{k}\cdot\vec{r})} + C.C.$$

$$= \hat{E}_\perp + \hat{E}_\perp^{Cl}(\alpha, t)$$

We also have $|\psi'(t)\rangle = \hat{T}_\alpha|\alpha(t)\rangle = |0\rangle$

Action of the unitary transformation $\hat{T}_\alpha(t)$

$$\hat{E}_\perp' = \hat{T}_\alpha(t)\hat{E}_\perp\hat{T}_\alpha(t)^+ = \hat{E}_\perp + E_\perp^{Cl}(\alpha, t)$$

$$|\psi'(t)\rangle = T_\alpha(t)|\alpha(t)\rangle = |0\rangle$$

$$\Downarrow$$

We can work with

$\hat{E}_\perp, |\alpha(t)\rangle$ **or** $\hat{E}_\perp + E_\perp^{Cl}(\alpha, t), |0\rangle$

Validates Semiclassical Optics for strong Coherent Fields!

Quantized Light – Matter Interactions

General Problem:

Quantum Atom ⟷ Quantized Modes Of EM Field

Electric Dipole Interaction

Starting Point: System Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{AF} \qquad (1)$$

$$\hat{H}_F = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}} + 1/2\right) \qquad \text{Field}$$

$$\hat{H}_A = \sum_i E_i |i\rangle\langle i| = \sum_i E_i \hat{\sigma}_{ii} \qquad \text{Atom}$$

$$\hat{H}_{AF} = -\hat{\vec{\mu}} \cdot \hat{\vec{E}}(\vec{r},t) \qquad \text{ED interaction}$$

E_i, $|i\rangle$: energies, energy levels of the atom

Dipole Operator:

$$\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij} |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij} \hat{\sigma}_{ij} \qquad (2)$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \hat{a}_{\vec{k}} u_{\vec{k}}(\vec{r}) + H.C., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\omega_{\vec{k}}}{2\varepsilon_0 V}}$$

2 polarization modes implicit

Pin down atom where $u_{\vec{k}}(\vec{r}) = 1$

– anywhere if $u_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

– if $u_{\vec{k}}(\vec{r}) = \sin(kz)$ then where $\sin(kz) = 1$

$$\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^+_{\vec{k}}\right) \qquad (3)$$

Quantized Light – Matter Interactions

Dipole Operator:

(2) $$\boxed{\hat{\vec{\mu}} = \sum_{i,j} \vec{\mu}_{ij}\, |i\rangle\langle j| = \sum_{i,j} \vec{\mu}_{ij}\, \hat{\sigma}_{ij}}$$

Field Operator:

$$\vec{E}(\vec{r},t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}}\, \mathcal{E}_{\vec{k}}\, \hat{a}_{\vec{k}}\, u_{\vec{k}}(\vec{r}) + H.c., \quad \mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar \omega_{\vec{k}}}{2\varepsilon_0 V}}$$

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(3) $$\boxed{\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}(t) = \sum_{\vec{k}} \vec{\varepsilon}_{\vec{k}}\, \mathcal{E}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right)}$$

Combining (2) & (3):

$$\hat{H}_{AF} = \sum_{ij} \sum_{\vec{k}} -\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\, \mathcal{E}_{\vec{k}}\, \hat{\sigma}_{ij} \left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right)$$

$$= \sum_{ij} \sum_{\vec{k}} \hbar\, g^{(ij)}_{\vec{k}} \left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right)$$

where $g^{(ij)}_{\vec{k}} = \dfrac{\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\, \mathcal{E}_{\vec{k}}}{\hbar}$

Rabi Freq., note sign convention

2-level atom ⇨ $(i,j) = (1,2)$ **:**

$$\boxed{\hat{H}_{AF} = \hbar \sum_{\vec{k}} \left(g_{\vec{k}}\, \hat{\sigma}_{21} + g^{*}_{\vec{k}}\, \hat{\sigma}_{12}\right)\left(\hat{a}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\right)}$$

Define:

$$\left.\begin{aligned} \hat{\sigma}_{+} &= \hat{\sigma}_{21} = |2\rangle\langle 1| \\ \hat{\sigma}_{-} &= \hat{\sigma}_{12} = |1\rangle\langle 2| \\ \hat{\sigma}_{z} &= \hat{\sigma}_{22} - \hat{\sigma}_{11} = |2\rangle\langle 2| - |1\rangle\langle 1| \end{aligned}\right\} \Rightarrow$$

Pauli matrices

$$\hat{\sigma}_{x} = \tfrac{1}{2}\left(\hat{\sigma}_{+} + \hat{\sigma}_{-}\right)$$

$$\hat{\sigma}_{y} = \tfrac{1}{2i}\left(\hat{\sigma}_{+} - \hat{\sigma}_{-}\right)$$

$$\hat{\sigma}_{z}$$

Quantized Light – Matter Interactions

Combining (2) & (3):

$$\boxed{\begin{aligned}\hat{H}_{AF} &= \sum_{ij}\sum_{\vec{k}} -\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}\,\hat{\sigma}_{ij}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})\\ &= \sum_{ij}\sum_{\vec{k}} \hbar\, g^{(ij)}_{\vec{k}}\,(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})\end{aligned}}$$

where $g^{(ij)}_{\vec{k}} = \dfrac{\vec{\mu}_{ij}\cdot\vec{\varepsilon}_{\vec{k}}\,\mathcal{E}_{\vec{k}}}{\hbar}$

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2-level atom ⇒ $(i,j)=(1,2)$:

$$\boxed{\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{21}+g^{*}_{\vec{k}}\hat{\sigma}_{12}\right)(\hat{a}_{\vec{k}}+\hat{a}^{+}_{\vec{k}})}$$

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$$\hat{\sigma}_{z}$$

With this notation **(4)**

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + \cancel{g_{\vec{k}}\hat{\sigma}_{+}\hat{a}^{+}_{\vec{k}}} + \cancel{g_{\vec{k}}\hat{\sigma}_{-}\hat{a}_{\vec{k}}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$

Energy conservation?

$$\hat{H}_{AF} = \hbar\sum_{\vec{k}}\left(g_{\vec{k}}\hat{\sigma}_{+}\hat{a}_{\vec{k}} + g_{\vec{k}}\hat{\sigma}_{-}\hat{a}^{+}_{\vec{k}}\right)$$