Quantum States of the Quantized Field

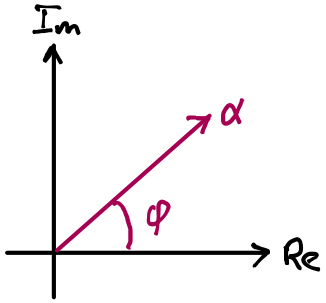
Amplitude and Phase

- Key characteristics of classical fields
- Need equivalents for quantum fields

Classical Field

$$E(z,t) = \mathcal{E}_k \boxed{\alpha} e^{-i(\omega t - kz)} + c.c.$$

α: $|\alpha| e^{i\varphi}$



Quantum Field

$$\hat{E}(z,t) = \mathcal{E}_k \boxed{\hat{a}} e^{-i(\omega t - kz)} + H.C.$$

$\hat{a}$: Non-Hermitian!

Separate in amplitude & phase?

Consider operators

$$\hat{a} = (\hat{N}+1)^{1/2} e^{\hat{}}(i\varphi)$$

$$\hat{a}^\dagger = e^{\hat{}}(-i\varphi)(\hat{N}+1)^{1/2}$$

"phase" "amplitude"

$$e^{\hat{}}(i\varphi) = (\hat{N}+1)^{-1/2}\hat{a}$$

$$e^{\hat{}}(-i\varphi) = \hat{a}^\dagger(\hat{N}+1)^{-1/2}$$

"Phase operators"

$$e^{\hat{}}(i\varphi)\, e^{\hat{}}(-i\varphi) = 1 \qquad e^{\hat{}}(i\varphi) = e^{\hat{}}(-i\varphi)^\dagger$$

$$e^{\hat{}}(-i\varphi)\, e^{\hat{}}(i\varphi) = 1 \qquad = \left[e^{\hat{}}(-i\varphi)\right]^{-1}$$

- Analogous to classical phases
- Non-Hermitian, NOT observables

Quadrature operators?

$$\cos\hat{\varphi} = \frac{1}{2}\left[e^{\hat{}}(i\varphi) + e^{\hat{}}(-i\varphi)\right]$$

$$= \frac{1}{2}\left[(\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^\dagger(\hat{N}+1)^{-1/2}\right]$$

$$\sin\hat{\varphi} = \frac{1}{2i}\left[e^{\hat{}}(i\varphi) - e^{\hat{}}(i\varphi)\right]$$

$$= \frac{1}{2i}\left[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^\dagger(\hat{N}+1)^{-1/2}\right]$$

- Hermitian -> observables
- but ultimately too cumbersome

Let's rewind and try again…

Quantum States of the Quantized Field

"Phase operators"

$$\hat{\exp}(i\varphi)\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$

$$\hat{\exp}(-i\varphi)\hat{\exp}(i\varphi) = 1 \qquad = \left[\hat{\exp}(-i\varphi)\right]^{-1}$$

- Analogous to classical phases
- Non-Hermitian, NOT observables

Quadrature operators?

$$\hat{\cos}\varphi = \frac{1}{2}\left[\hat{\exp}(i\varphi) + \hat{\exp}(-i\varphi)\right]$$
$$= \frac{1}{2}\left[(\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

$$\hat{\sin}\varphi = \frac{1}{2i}\left[\hat{\exp}(i\varphi) - \hat{\exp}(i\varphi)\right]$$
$$= \frac{1}{2i}\left[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

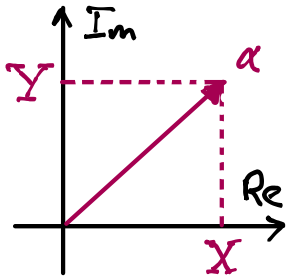
- Hermitian -> observables
- but ultimately too cumbersome

Let's rewind and try again…

Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k a_k(t)}\, e^{ikz} + c.c.$$

complex amplitude for mode e^{ikz}



Define

$$X(t) = \mathrm{Re}\left[\alpha_k(t)\right] = \frac{1}{2}\left[\alpha_k(t) + \alpha_k^{*}(t)\right] = Q(t)$$

$$Y(t) = \mathrm{Im}\left[\alpha_k(t)\right] = \frac{1}{2i}\left[\alpha_k(t) - \alpha_k^{*}(t)\right] = P(t)$$

Quantization: $\alpha \to \hat{a}$, $\alpha^{*} \to \hat{a}^{+}$

$$\left.\begin{aligned}\hat{X}(t) &= \frac{1}{2}\left[\hat{a}_k(t) + \hat{a}_k^{+}(t)\right] = \hat{Q}(t)\\ \hat{Y}(t) &= \frac{1}{2i}\left[\hat{a}_k(t) - \hat{a}_k^{+}(t)\right] = \hat{P}(t)\end{aligned}\right\} \quad \left[\hat{X}(t), \hat{Y}(t)\right] = i/2$$

$$\hat{E}(z,t) = \mathcal{E}_k\left(\hat{X}(t) + i\hat{Y}(t)\right)e^{ikz} + H.C.$$
$$= \mathcal{E}_k\left[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)\right]$$

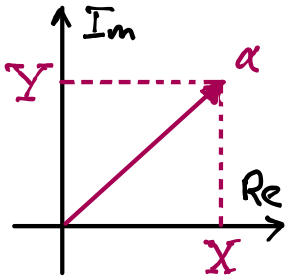
– same info, easier to work with –

Quantum States of the Quantized Field

Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k a_k(t)}\, e^{ikz} + c.c.$$

complex amplitude for mode e^{ikz}



Define

$$X(t) = \mathrm{Re}[\alpha_k(t)] = \frac{1}{2}[\alpha_k(t) + \alpha_k^*(t)] = Q(t)$$

$$Y(t) = \mathrm{Im}[\alpha_k(t)] = \frac{1}{2i}[\alpha_k(t) - \alpha_k^*(t)] = P(t)$$

Quantization: $\alpha \to \hat{a}$, $\alpha^* \to \hat{a}^+$

$$\hat{X}(t) = \frac{1}{2}[\hat{a}_k(t) + \hat{a}_k^+(t)] = \hat{Q}(t)$$

$$\hat{Y}(t) = \frac{1}{2i}[\hat{a}_k(t) - \hat{a}_k^+(t)] = \hat{P}(t)$$

$$[\hat{X}(t), \hat{Y}(t)] = i/2$$

$$\hat{E}(z,t) = \mathcal{E}_k(\hat{X}(t) + i\hat{Y}(t))e^{ikz} + H.C.$$

$$= \mathcal{E}_k[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)]$$

– same info, easier to work with –

Quantum States of the Field in Mode k

Number States (Foch states)

$$\hat{a}^+\hat{a}\,|n\rangle = n\,|n\rangle$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \frac{1}{2}(n + 1/2)$$

$$\Delta X \Delta Y = \frac{1}{2}(n + 1/2)$$

– HIGHLY non-classical, $\langle \hat{E} \rangle = 0$

– VERY hard to make for large n

Quantum States of the Quantized Field

Quantum States of the Field in Mode k

Number States (Foch states)

$$\hat{a}^\dagger \hat{a} |n\rangle = n|n\rangle$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \frac{1}{2}(n+1/2)$$

$$\Delta X \Delta Y = \frac{1}{2}(n+1/2)$$

– HIGHLY non-classical, $\langle \hat{E} \rangle = 0$

– VERY hard to make for large n

Coherent States (Quasi-classical states)

– Closest approximation to classical field

– See Cohen-Tannoudj, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle \hat{X}(t) \rangle = \langle \psi|\hat{X}(t)|\psi\rangle = X(t), \quad \langle \hat{Y}(t) \rangle = Y(t)$$

$$\langle \hat{H}(t) \rangle = \hbar\omega(|\alpha(t)|^2 + 1/2)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}$$

equivalently

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

(1) $\langle \hat{a}(0) \rangle = \langle \psi|\hat{a}(0)|\psi\rangle = \alpha(0)$

(2) $\langle \hat{a}^\dagger(0)\hat{a}(0) \rangle = \alpha(0)^*\alpha(0)$

Quantum States of the Quantized Field

Coherent States (Quasi-classical states)

– Closest approximation to classical field

– See Cohen-Tannoudj, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle\hat{X}(t)\rangle = \langle\psi|\hat{X}(t)|\psi\rangle = X(t), \quad \langle\hat{Y}(t)\rangle = Y(t)$$

$$\langle\hat{H}(t)\rangle = \hbar\omega\left(|\alpha(t)|^2 + 1/2\right)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^+(t) = \hat{a}^+(0)e^{i\omega t}$$

equivalently

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

(1) $\langle\hat{a}(0)\rangle = \langle\psi|\hat{a}(0)|\psi\rangle = \alpha(0)$

(2) $\langle\hat{a}^+(0)\hat{a}(0)\rangle = \alpha(0)^*\alpha(0)$

Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

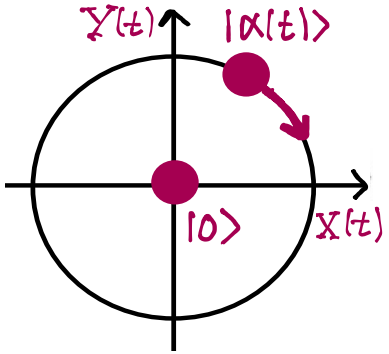
Physical properties

$$\langle\hat{X}(t)\rangle = \mathrm{Re}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\langle\hat{Y}(t)\rangle = \mathrm{Im}\left[\alpha(0)e^{-i\omega t}\right]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$

$$\Delta X\,\Delta Y = 1/4$$



Quantum States of the Quantized Field

Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

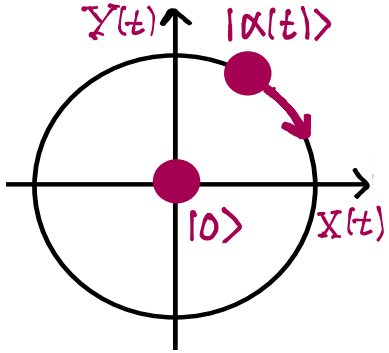
Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Physical properties

$$\langle \hat{X}(t) \rangle = \mathrm{Re}[\alpha(0) e^{-i\omega t}]$$
$$\langle \hat{Y}(t) \rangle = \mathrm{Im}[\alpha(0) e^{-i\omega t}]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$
$$\Delta X \Delta Y = 1/4$$



Photon statistics

Measure $\hat{N}$ ⇒ outcomes n

$$P(n) = \langle \alpha | n \rangle \langle n | \alpha \rangle = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

Poisson distribution w/ mean $\bar{n} = |\alpha|^2$; variance $\Delta n^2 = |\alpha|^2$

$$\Delta n = \sqrt{\bar{n}}$$ – Shot Noise

Quantum States of the Quantized Field

Begin Lecture 04-12-2021

Photon statistics

Measure $\hat{N}$ ⇒ { outcomes n ; $P(n) = \langle\alpha|n\rangle\langle n|\alpha\rangle = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$

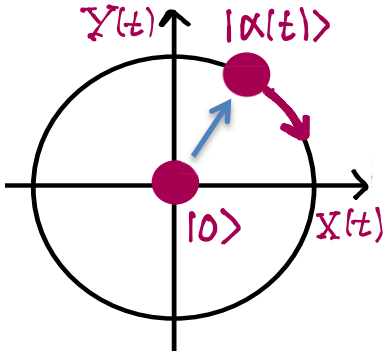
⇓

Poisson distribution w/ { mean $\bar{n} = |\alpha|^2$; variance $\Delta n^2 = |\alpha|^2$

⇓

$$\Delta n = \sqrt{\bar{n}} \quad \text{– Shot Noise}$$

More about Coherent States



⇒ Coherent States as translated Vacuum States?

Generating Coherent States from the Vacuum

Definition: $\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$

↑ Unitary, equals translation

Glaubers formula (from BCH formula)

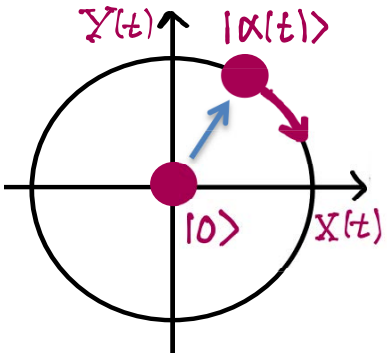
$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

End Lecture 04-09-2021

Quantum States of the Quantized Field

More about Coherent States



Coherent States as translated Vacuum States?

Generating Coherent States from the Vacuum

Definition: $\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$

Unitary, equals translation

Glaubers formula (from BCH formula)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Apply to $[\alpha\hat{a}^+, -\alpha^*\hat{a}] = \alpha^*\alpha$

($\alpha\hat{a}^+ \to \hat{A}$, $-\alpha^*\hat{a} \to \hat{B}$, $\alpha^*\alpha \to [\hat{A},\hat{B}]$)

$$\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+} e^{-\alpha\hat{a}}$$

Remember: $\hat{a}|0\rangle = 0$

$$e^{-\alpha^*\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^*\hat{a})^n}{n!}|0\rangle = |0\rangle$$

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+}|0\rangle$$

$$= e^{-|\alpha|^2} \sum_n \frac{(\alpha\hat{a}^+)^n}{n!}|0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

Quantum States of the Quantized Field

Apply to $[\alpha\hat{a}^+, -\alpha^*\hat{a}] = \alpha^*\alpha$

$\hat{A}$ $\hat{B}$ $[\hat{A},\hat{B}]$

$$\hat{D}(\alpha) = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^+} e^{-\alpha\hat{a}}$$

Remember: $\hat{a}|0\rangle = 0$

$$e^{-\alpha^*\hat{a}}|0\rangle = \sum_n \frac{(-\alpha^*\hat{a})^n}{n!}|0\rangle = |0\rangle$$

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}\, e^{\alpha\hat{a}^+}|0\rangle$$

$$= e^{-|\alpha|^2} \sum_n \frac{(\alpha\hat{a}^+)^n}{n!}|0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat{a}^+ - \alpha^*\hat{a} = (\alpha-\alpha^*)\hat{X} + i(\alpha+\alpha^*)\hat{Y}$$

$$= i2Y\hat{X} + i2X\hat{Y}$$

where $X = \langle\alpha|\hat{X}|\alpha\rangle$, $Y = \langle\alpha|\hat{Y}|\alpha\rangle$

Glaubers formula again:

$$\hat{D}(\alpha) = e^{i2Y\hat{X} + i2X\hat{Y}} = e^{-XY/4}\, e^{i2Y\hat{X}}\, e^{i2X\hat{Y}}$$

Recall: $\hat{S}(q) = e^{-iq\hat{p}/\hbar}$ ➡ **translation by** q

$\hat{S}(p) = e^{-ip\hat{q}/\hbar}$ ➡ **translation by** p

where $q = q_0 X$, $p = p_0 Y$

$\hat{q} = q_0\hat{X}$, $\hat{p} = p_0\hat{Y}$

& $X_0 p_0 = 2\hbar$

Quantum States of the Quantized Field

OK – $\hat{D}(\alpha)$ generates $|\alpha\rangle$ from the vacuum!

Rewrite:

$$\alpha\hat{a}^{+} - \alpha^{*}\hat{a} = (\alpha-\alpha^{*})\hat{X} + i(\alpha+\alpha^{*})\hat{Y}$$
$$= i2Y\hat{X} + i2X\hat{Y}$$

where $X = \langle\alpha|\hat{X}|\alpha\rangle$, $Y = \langle\alpha|\hat{Y}|\alpha\rangle$

Glaubers formula again:

$$\hat{D}(\alpha) = e^{i2Y\hat{X}+i2X\hat{Y}} = e^{-XY/4}\, e^{i2Y\hat{X}}\, e^{i2X\hat{Y}}$$

Recall: $\hat{S}(q) = e^{-iq\hat{P}/\hbar}$ ⇨ **translation by** q

Recall: $\hat{S}(p) = e^{-ip\hat{q}/\hbar}$ ⇨ **translation by** p

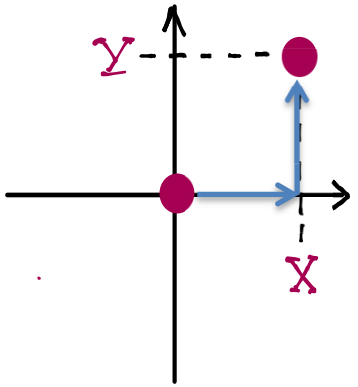
where $q = q_0 X$, $P = p_0 Y$

$\hat{q} = q_0\hat{X}$, $\hat{P} = p_0\hat{Y}$ & $X_0 p_0 = 2\hbar$

This gives us

$$\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}$$

$\hat{D}(\alpha)$ **translates along** X **then** Y



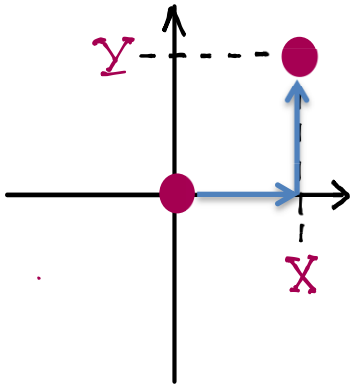
Discussion –
How to do this?

Quantum States of the Quantized Field

This gives us

$$\hat{S}(q) = \hat{S}(X) = e^{i2X\hat{Y}}, \quad \hat{S}(p) = \hat{S}(Y) = e^{i2Y\hat{X}}$$

$\hat{D}(\alpha)$ **translates along** X **then** Y



Discussion –
How to do this?

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t) \quad @ \; t = 0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k (\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\hat{H} = \hbar\omega_k(\hat{a}^+\hat{a} + 1/2) + \hbar\lambda(t)(\hat{a} + \hat{a}^+)$$

$$\lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)$$

Homework Problem (voluntary)

$$\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega_k)T/2} \frac{\sin[(\omega-\omega_k)T/2]}{(\omega-\omega_k)/2}$$

Quantum States of the Quantized Field

Coherent States from Classical Dipole Radiation

Classical Dipole $d(t) = d_0 \cos(\omega t) \quad @\ t = 0$

Quantized Field $\hat{E}(z) = \mathcal{E}_k (\hat{a} + \hat{a}^+)$

Dipole-Field Interaction

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + 1/2\right) + \hbar\lambda(t)\left(\hat{a} + \hat{a}^+\right)$$

$$\lambda(t) = -\frac{d(t)\mathcal{E}_k}{\hbar} = \lambda_0 \cos(\omega t)$$

Homework Problem (voluntary)

$$\alpha(T) = -i\frac{\lambda_0}{2} e^{-i(\omega-\omega_k)T/2} \frac{\sin[(\omega-\omega_k)T/2]}{(\omega-\omega_k)/2}$$

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ drives $\hat{E}(t)$

classical dipole + quantum fluctuations

coherent state + quantum fluctuations

Drive from $0 < t < T$

$$\alpha(t) = \alpha(T) e^{-i\omega_k(t-T)}$$

Quantum States of the Quantized Field

Recall from Semi-Classical Laser Theory

$\langle \hat{p}(t) \rangle$ drives $\hat{E}(t)$

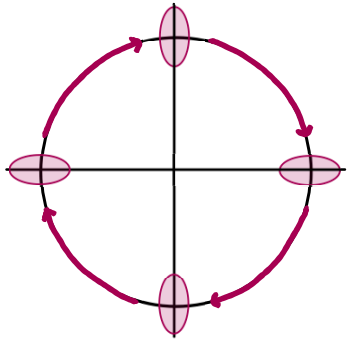
classical dipole	coherent state
+ quantum fluctuations	+ quantum fluctuations

Squeezed States

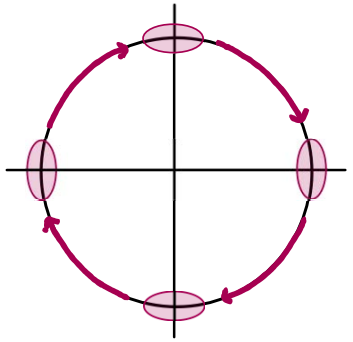
Minimum uncertainty states w/asymmetry

$$\Delta X \, \Delta Y = 1/4 , \quad \Delta X(t) \neq \Delta Y(t)$$

Phase Squeezing



Amplitude Squeezing



Requires interaction with Nonlinear medium