Quantum States of the Quantized Field

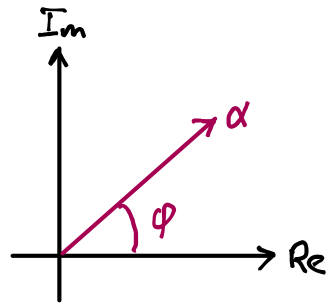
Amplitude and Phase

– Key characteristics of classical fields
– Need equivalents for quantum fields

Classical Field

$$E(z,t) = \mathcal{E}_k \boxed{\alpha} e^{-i(\omega t - kz)} + c.c.$$

$|\alpha| e^{i\varphi}$



Quantum Field

$$\hat{E}(z,t) = \mathcal{E}_k \boxed{\hat{a}} e^{-i(\omega t - kz)} + H.C.$$

Non-Hermitian!
Separate in amplitude & phase?

Consider operators

$$\hat{a} = (\hat{N}+1)^{1/2} \hat{\exp}(i\varphi)$$
$$\hat{a}^{+} = \hat{\exp}(-i\varphi)(\hat{N}+1)^{1/2}$$

"phase" "amplitude"

$$\hat{\exp}(i\varphi) = (\hat{N}+1)^{-1/2} \hat{a}$$
$$\hat{\exp}(-i\varphi) = \hat{a}^{+}(\hat{N}+1)^{-1/2}$$

"Phase operators"

$$\hat{\exp}(i\varphi)\hat{\exp}(-i\varphi) = 1 \qquad \hat{\exp}(i\varphi) = \hat{\exp}(-i\varphi)^{+}$$
$$\hat{\exp}(-i\varphi)\hat{\exp}(i\varphi) = 1 \qquad = \left[\hat{\exp}(-i\varphi)\right]^{-1}$$

– Analogous to classical phases
– Non-Hermitian, NOT observables

Quadrature operators?

$$\hat{\cos}\varphi = \frac{1}{2}\left[\hat{\exp}(i\varphi) + \hat{\exp}(-i\varphi)\right]$$
$$= \frac{1}{2}\left[(\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

$$\hat{\sin}\varphi = \frac{1}{2i}\left[\hat{\exp}(i\varphi) - \hat{\exp}(i\varphi)\right]$$
$$= \frac{1}{2i}\left[(\hat{N}+1)^{-1/2}\hat{a} - \hat{a}^{+}(\hat{N}+1)^{-1/2}\right]$$

– Hermitian -> observables
– but ultimately too cumbersome

Let's rewind and try again…

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"Phase operators"

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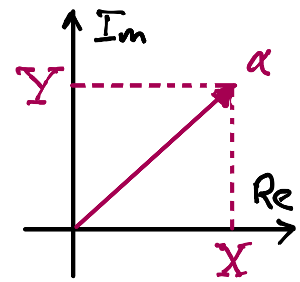
Quadratures of the Classical Field – Take Two

$$E(z,t) = \underbrace{\mathcal{E}_k \alpha_k(t)}_{\text{complex amplitude for mode } e^{ikz}} e^{ikz} + c.c.$$



Define

$$\boxed{\begin{aligned} X(t) &= Re\left[\alpha_k(t)\right] = \frac{1}{2}\left[\alpha_k(t) + \alpha_k^{*}(t)\right] = Q(t) \\ Y(t) &= Im\left[\alpha_k(t)\right] = \frac{1}{2i}\left[\alpha_k(t) - \alpha_k^{*}(t)\right] = P(t) \end{aligned}}$$

Quantization: $\alpha \to \hat{a}, \quad \alpha^{*} \to \hat{a}^{+}$

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– same info, easier to work with –

Quantum States of the Quantized Field

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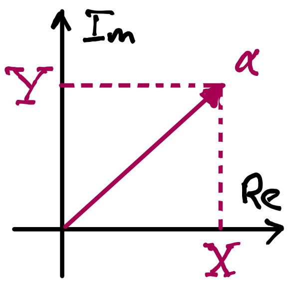

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$$\hat{E}(z,t) = \mathcal{E}_k(\hat{X}(t) + i\hat{Y}(t))e^{ikz} + H.C.$$

$$= \mathcal{E}_k[\hat{X}(t)\cos(kz) - \hat{Y}(t)\sin(kz)]$$

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Quantum States of the Field in Mode k

Number States (Foch states)

$$\boxed{\hat{a}^+\hat{a}|n\rangle = n|n\rangle}$$

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \tfrac{1}{2}(n + 1/2)$$

$$\boxed{\Delta X \Delta Y = \tfrac{1}{2}(n + 1/2)}$$

– **HIGHLY non-classical,** $\langle \hat{E} \rangle = 0$

– **VERY hard to make for large** n

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Coherent States (Quasi-classical states)

– Closest approximation to classical field

– See Cohen-Tannoudj, complement G_V

Definition: $|\psi\rangle$ is coherent (quasiclassical) iff

$$\langle \hat{X}(t)\rangle = \langle \psi|\hat{X}(t)|\psi\rangle = X(t), \quad \langle \hat{Y}(t)\rangle = Y(t)$$

$$\langle \hat{H}(t)\rangle = \hbar\omega\left(|\alpha(t)|^2 + 1/2\right)$$

noting that

$$\hat{X}(t) \propto \hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{Y}(t) \propto \hat{a}^{+}(t) = \hat{a}^{+}(0)e^{i\omega t}$$

equivalently

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Cohen-Tannoudji, Lecture Notes

equivalently

Definition: a state $|\alpha\rangle$ is coherent iff

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

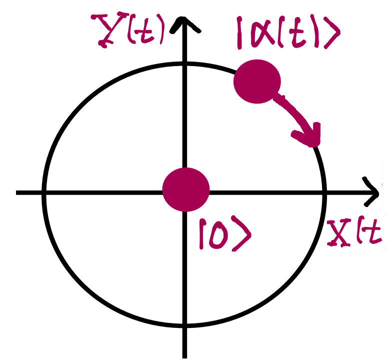
Finally, one can show

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Physical properties

$$\langle \hat{X}(t)\rangle = Re\left[\alpha(0)e^{-i\omega t}\right]$$
$$\langle \hat{Y}(t)\rangle = Im\left[\alpha(0)e^{-i\omega t}\right]$$

$$\Delta X(t) = \Delta Y(t) = 1/2$$
$$\Delta X\,\Delta Y = 1/4$$



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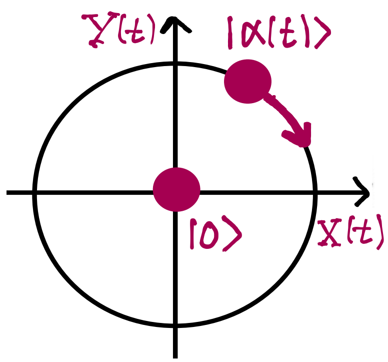

Photon statistics

Measure $\hat{N}$ ⇨
- outcomes n
- $P(n) = \langle\alpha|n\rangle\langle n|\alpha\rangle = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}$

Poisson distribution w/
- **mean** $\bar{n} = |\alpha|^2$
- **variance** $\Delta n^2 = |\alpha|^2$

$$\Delta n = \sqrt{\bar{n}}$$ **– Shot Noise**