Quantum Electrodynamics – QED

Starting point: Maxwells Equations

(1) $\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$

(2) $\nabla \cdot \vec{B}(\vec{r},t) = 0$

(3) $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$

(4) $\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

Implicit: Charges & Fields in Vacuum
No "medium response"

Same issue as with our introductory example:

Maxwells eqs are non-local

We need to put the classical description in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

(1) $i\vec{k} \cdot \vec{E}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k} \cdot \vec{B}(\vec{k},t) = 0$

(3) $i\vec{k} \times \vec{E}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{k},t)$

(4) $i\vec{k} \times \vec{B}(\vec{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{k},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{k},t)$

Fourier Transform:
$$\left\{\begin{array}{l} \nabla \cdot \vec{G} \rightleftarrows i\vec{k} \cdot \vec{G} \\ \nabla \times \vec{G} \rightleftarrows i\vec{k} \times \vec{G} \end{array}\right.$$

Note: This is a Normal Mode decomposition

No charges -> No coupling between modes with different $\vec{k}$

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Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t)=\vec{\mathcal{E}}_{\parallel}(\vec{k},t)+\vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{\mathcal{B}}(\vec{k},t)=\cancel{\vec{\mathcal{B}}_{\parallel}(\vec{k},t)}+\vec{\mathcal{B}}_{\perp}(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

Note: $\left\{ \begin{array}{l} -\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}} \text{ is the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \\ \vec{\mathcal{E}}_{\parallel} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \end{array} \right.$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel}=\frac{\vec{k}}{k}\mathcal{E}_{\parallel}=\frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right)=-i\frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

Coulomb field from the charges

Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>

Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

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Entirely Transverse

Note:
- $-\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}$ is the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$
- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k}$ × the projection of $\vec{\mathcal{E}}$ onto $\vec{k}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k} i\vec{k}\cdot\vec{\mathcal{E}}\right) = -i\frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

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Eqs for Transverse Fields, from MEqs (3) & (4)

(5a) $\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_{\perp}(\vec{k},t)$

(6a) $\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$

(3) $i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t)$

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inverse FT

(5b) $\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_{\perp}(\vec{r},t)$

(6b) $\frac{\partial}{\partial t}\vec{E}_{\perp}(\vec{r},t) = c^2\nabla\times\vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{r},t)$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_{\perp}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t^2}\vec{j}_{\perp}(\vec{r},t)$$

Quantum Electrodynamics – QED

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Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$

Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\varepsilon}_x E_x(z,t)$ and expand

fiducial mass

(7) $E_x(z,t) = \sum_j A_j q_j(t)\sin(k_j z)$, $A_j = \sqrt{\frac{\omega_j^2 m_j}{2\varepsilon_0 V}}$

MEq (4) w/no charges

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\varepsilon}_x\frac{1}{c^2}\sum_j A_j\dot{q}_j(t)\sin(k_j t)$$

(4) $\nabla\times\vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

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$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}_\perp(\vec{r},t)$$

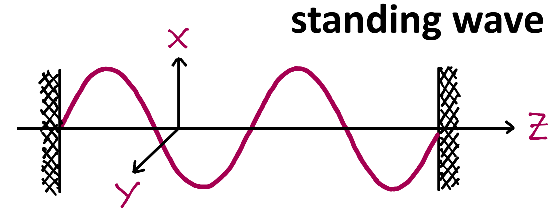
Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$

standing wave



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$$= \vec{\varepsilon}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right) = -\vec{\varepsilon}_x\frac{\partial B_y}{\partial z}$$

$\vec{B}$ transverse ⇒ $B_z = 0$

Quantum Electrodynamics – QED

Normal Modes in a 1D Cavity

Length L

Cross section A

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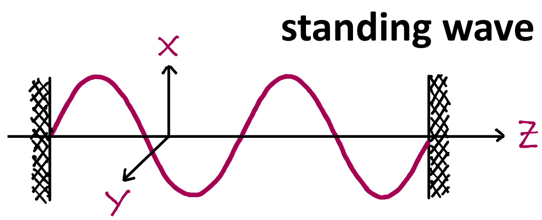

Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\mathcal{E}}_x\, E_x(z,t)$ and expand

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$$\boxed{(7)\quad E_x(z,t) = \sum_j A_j\, q_j(t) \sin(k_j z)\,, \quad A_j = \sqrt{\frac{\omega_j^2 m_j}{2\varepsilon_0 V}}}$$

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B transverse $\Rightarrow$ $B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\mathcal{E}}_z \;\Rightarrow\; \vec{B}(z,t) = \vec{\mathcal{E}}_y\, B_y(z,t)$$

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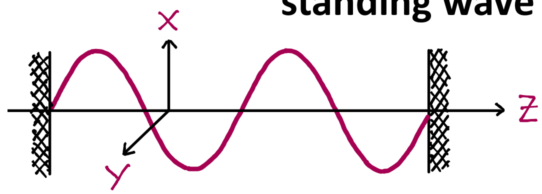

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Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j \frac{A_j}{c^2} \dot{q}_j(t) \sin(k_j z)$$

$$\boxed{(8)\quad B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\, \dot{q}_j(t) \cos(k_j z)}$$

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\varepsilon_0 A}{2}\int_0^L dz\left(|\vec{E}|^2 + c^2|\vec{B}|^2\right) =$$

$$\frac{\varepsilon_0 A}{2}\int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j}\dot{q}_j(t)^2 \cos^2(k_j z)\right]$$

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$$\frac{\varepsilon_0 A}{2} \int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j} \dot{q}_j(t)^2 \cos^2(k_j z) \right]$$

Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{\omega_j^2 m_j}{2 \varepsilon_0 V}$ we finally get

$$\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]$$

Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\varepsilon_0 A}{2} \int_0^L dz \left(c^2 |\vec{B}|^2 - |\vec{E}|^2 \right)$$

$$= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

☑

Check $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_\perp(\vec{r}, t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

Quantum Electrodynamics – QED

Integrating over the Cavity volume

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Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\varepsilon_0 A}{2}\int_0^L dz\left(c^2|\vec{B}|^2 - |\vec{E}|^2\right)$$

$$= \sum_j \left[\tfrac{1}{2} m_j \dot{q}_j^2 - \tfrac{1}{2} m_j \omega_j^2 q_j^2\right]$$

Check $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

And Finally:

$$\boxed{p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j}$$

As before, a collection of Harmonic Oscillators, ready for quantization!

End 03-31-2021

Begin 04-05-2021

Quantum Electrodynamics – QED

Todays Jump-off Point

$$E_x(z,t) = \sum_j A_j\, q_j(t) \sin(k_j z)$$

$$A_j = \sqrt{\frac{\omega_j^2 m_j}{2\varepsilon_0 V}}$$

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\, \dot{q}_j(t) \cos(k_j z)$$

$$\mathcal{L} = \frac{\varepsilon_0 A}{2}\int_0^L dz \left(c^2|\vec{B}|^2 - |\vec{E}|^2\right) = \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j q_j^2\right]$$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j$$

$$\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2\right]$$

Classical Fields

Dimensionless Field Variables:

$$Q_j = q_j / q_{0,j}, \quad q_{0,j} = \sqrt{2\hbar / m_j \omega_j}$$

$$P_j = p_j / p_{0,j}, \quad p_{0,j} = \sqrt{2\hbar m_j \omega_j}$$

$$\boxed{\alpha_j(t) = Q_j(t) + i\, P_j(t) = \alpha_j(0)\, e^{-i\omega_j t}}$$

$$E_x(z,t) = \sum_j A_j\, q_j(t) \sin(k_j z), \quad \mathcal{E}_j = A_j q_{0,j} = \sqrt{\frac{\hbar\omega_j}{\varepsilon_0 V}}$$

$$= \sum_j \mathcal{E}_j \left[\alpha_j(t) + \alpha_j^*(t)\right] \sin(k_j z)$$

field "per photon"

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\, \dot{q}_j(t) \cos(k_j z)$$

$$= -\frac{i}{c} \sum_j \mathcal{E}_j \left[\alpha_j(t) - \alpha_j^*(t)\right] \cos(k_j t)$$

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$$= -\frac{i}{c} \sum_j \mathcal{E}_j \left[\alpha_j(t) - \alpha_j^*(t)\right] \cos(k_j t)$$

Standard Quantization Procedure

$$q_j \to \hat{q}_j\,, \quad p_j \to \hat{p}_j\,, \qquad [\hat{q}_j, \hat{p}_{j'}] = i\hbar\,\delta_{jj'}$$

$$\alpha_j(t) \to \hat{a}_j\,, \quad \alpha_j^*(t) \to \hat{a}_j^\dagger\,, \qquad [\hat{a}_j, \hat{a}_{j'}^\dagger] = \delta_{jj'}$$

$$\hat{E}_x(z) = \sum_j \mathcal{E}_j (\hat{a}_j + \hat{a}_j^\dagger) \sin(k_j z)$$

$$\hat{B}_y(z) = -\frac{i}{c} \sum_j \mathcal{E}_j (\hat{a}_j - \hat{a}_j^\dagger) \cos(k_j z)$$

Total Field

$$\hat{\vec{E}}(z) = \vec{\varepsilon}_x \hat{E}_x(z) + \vec{\varepsilon}_y \hat{E}_y(z)$$

$$\hat{\vec{B}}(z) = \vec{\varepsilon}_x \hat{B}_x(z) + \vec{\varepsilon}_y \hat{B}_y(z)$$

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$$E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \quad \mathcal{E}_j = A_j q_{0,j} = \sqrt{\frac{\hbar\omega_j}{\varepsilon_0 V}}$$

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Standard Quantization Procedure

$$q_j \to \hat{q}_j, \quad p_j \to \hat{p}_j, \qquad [\hat{q}_j, \hat{p}_{j'}] = i\hbar\,\delta_{jj'}$$

$$\alpha_j(t) \to \hat{a}_j, \quad \alpha_j^*(t) \to \hat{a}_j^\dagger, \qquad [\hat{a}_j, \hat{a}_{j'}^\dagger] = \delta_{jj'}$$

$$\hat{E}_x(z) = \sum_j \mathcal{E}_j (\hat{a}_j + \hat{a}_j^\dagger) \sin(k_j z)$$

$$\hat{B}_y(z) = -\frac{i}{c} \sum_j \mathcal{E}_j (\hat{a}_j - \hat{a}_j^\dagger) \cos(k_j z)$$

Total Field

$$\hat{\vec{E}}(z) = \vec{\varepsilon}_x \hat{E}_x(z) + \vec{\varepsilon}_y \hat{E}_y(z)$$

$$\hat{\vec{B}}(z) = \vec{\varepsilon}_x \hat{B}_x(z) + \vec{\varepsilon}_y \hat{B}_y(z)$$

Quantum Electrodynamics – QED

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Note:

These are the Field Operators in the Schrödinger Picture (*t*-dependence in states)

Often advantageous to use Heisenberg Picture (*t*-dependence in operators)

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Field Quantization in Free Space:

Normal Modes : $u_{\vec{k},\lambda}(\vec{r}) = \vec{\mathcal{E}}_{\vec{k},\lambda} e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$

Finite quantization volume: $\mathcal{E}_{\vec{k}} = \sqrt{\hbar\omega_j / 2\varepsilon_0 V}$

$L \times L \times L$

L large -> nature of boundary conditions not important

Periodic boundary conditions

$$|\vec{k}| = n 2\pi / L$$

Quantum Electrodynamics – QED

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$$\vec{E}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

λ: polarization index

$$\vec{B}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \frac{\vec{k}\times\vec{\varepsilon}_{\vec{k},\lambda}}{kc}\, \mathcal{E}_{\vec{k},\lambda}\, \alpha_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + c.c.$$

Quantization:

$$\alpha_{\vec{k},\lambda} \rightarrow \hat{a}_{\vec{k},\lambda}, \qquad [\hat{a}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}] = \delta_{\vec{k},\vec{k}'}\,\delta_{\lambda,\lambda'}$$

$$\alpha^{*}_{\vec{k},\lambda} \rightarrow \hat{a}^{+}_{\vec{k},\lambda}, \qquad [\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}] = [\hat{a}^{+}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}] = 0$$

$$\hat{\vec{E}}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})} + H.C.$$

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– Heisenberg Picture –

Quantum Electrodynamics – QED

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– Heisenberg Picture –

Positive & Negative Frequency Components:

$$\hat{\vec{E}}_\perp(\vec{r},t) = \hat{E}^{(+)}(\vec{r},t) + \hat{E}^{(-)}(\vec{r},t)$$

$$\hat{E}^{(+)}(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda}\, \mathcal{E}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}\, e^{-i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})}$$

$$\hat{E}^{(-)}(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\varepsilon}^*_{\vec{k},\lambda}\, \mathcal{E}^*_{\vec{k},\lambda}\, \hat{a}^+_{\vec{k},\lambda}\, e^{i(\omega_{\vec{k},\lambda} t - \vec{k}\cdot\vec{r})}$$

Wrap Up:

Read page 13 in handwritten Note Set for brief discussion of different, equivalent ways to put the QED formalism together, e. g.

$$\hat{E}_x \propto (\hat{a}_j + \hat{a}^+_j) \quad \& \quad \hat{B}_y \propto (\hat{a}_j - \hat{a}^+_j)$$

vs

$$\hat{E}_x \propto (\hat{a}_j - \hat{a}^+_j) \quad \& \quad \hat{B}_y \propto (\hat{a}_j + \hat{a}^+_j)$$

Quantum Electrodynamics – QED

Positive & Negative Frequency Components:

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vs

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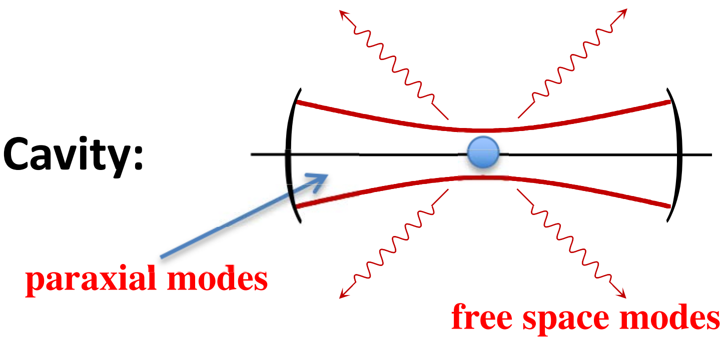
Other Normal Modes Sets

Atom in Cavity:



Wavepackets: (Milloni & Eberly, Sec. 12.8, p 381) (QED lecture notes, p 16)

Classical field pulse envelope

$$\vec{E}(\vec{r},t) = \vec{\varepsilon}\,\mathcal{E}_0\, u(z-ct)\, e^{i(k_0 z - \omega_0 t)} + c.c.$$

Mode volume $V = \int d^3r\, |u(x,y,z-ct)|^2$

Quantization $\mathcal{E}_0 \to \mathcal{E}_k \alpha_k \to \mathcal{E}_k \hat{a}_k$ etc.

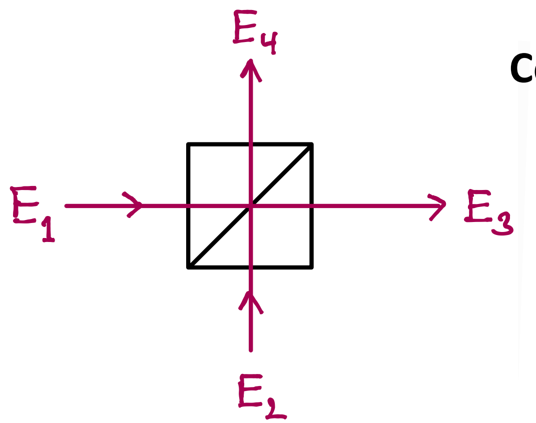
Wave-Particle Duality similar for Photons and Phonons

End 04-05-2021

Begin 04-07-2021

Example: Classical & Quantum Beamsplitters

Classical Beamsplitter



Coupled H & V modes

Linear symmetric input-output map

$$E_3 = tE_1 + rE_2$$
$$E_4 = rE_1 + tE_2$$

Energy conservation requires

$$|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$$

Choose $E_1 = 1,\ E_2 = 0$ ➡

$$|E_3|^2 + |E_4|^2 = |t|^2 + |r|^2 = 1$$

Choose $E_1 = \frac{1}{\sqrt{2}},\ E_2 = \frac{1}{\sqrt{2}}$ ➡

$$|E_3|^2 + |E_4|^2 = \frac{1}{2}|t+r|^2 =$$
$$|t|^2 + |r|^2 + tr^* + rt^* = 1$$

From this it follows that

$$\boxed{\begin{aligned} |t|^2 + |r|^2 &= 1 \\ tr^* + rt^* &= 0 \end{aligned}}$$

Classical input-output map

$$\boxed{\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}}$$

Quantum Beamsplitter

Heisenberg Picture ➡ **Field Operators obey Maxwells Eqs**

Classical field	**Quantum equivalent**
$E_\perp(\vec{r},t) \propto \alpha(t)$	$\hat{E}_\perp^{(+)}(\vec{r},t) \propto \hat{a}(t)$

Example: Classical & Quantum Beamsplitters

From this it follows that

$$\boxed{\begin{aligned} |t|^2+|r|^2 &= 1 \\ tr^*+rt^* &= 0 \end{aligned}}$$

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Quantum Beamsplitter

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$$E_\perp(\vec{r},t) \propto \alpha(t)$$

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$$\hat{E}_\perp^{(+)}(\vec{r},t) \propto \hat{a}(t)$$

Quantum Beamsplitter

$$\begin{pmatrix} \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}$$

Quantum input-output map

$$\boxed{\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}}$$

Invert Map

$$\begin{aligned} \hat{a}_3 &= t\hat{a}_1 + r\hat{a}_2 \\ \hat{a}_4 &= r\hat{a}_1 + t\hat{a}_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \hat{a}_1 &= t^*\hat{a}_3 + r^*\hat{a}_4 \\ \hat{a}_2 &= r^*\hat{a}_3 + t^*\hat{a}_4 \end{aligned}$$

Switch to creation operators ⇒

$$\boxed{\begin{aligned} \hat{a}_1^+ &= t\hat{a}_3^+ + r\hat{a}_4^+ \\ \hat{a}_2^+ &= r\hat{a}_3^+ + t\hat{a}_4^+ \end{aligned}}$$

Example: Classical & Quantum Beamsplitters

Quantum Beamsplitter

$$\begin{pmatrix} \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}$$

Quantum input-output map

$$\boxed{\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}}$$

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$$\hat{a}_3 = t\hat{a}_1 + r\hat{a}_2 \qquad \hat{a}_1 = t^*\hat{a}_3 + r^*\hat{a}_4$$
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Switch to Schrödinger Picture

General input state: 2-mode vacuum

$$\boxed{|\psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}}(\hat{a}_1^+)^n g_m \frac{1}{\sqrt{m!}}(\hat{a}_2^+)^m |0\rangle}$$

The BS maps $\hat{a}_1^+, \hat{a}_2^+$ to linear combinations of $\hat{a}_3^+, \hat{a}_4^+$

General output state: (Schrödinger Picture)

$$\boxed{|\psi_{out}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}}(t\hat{a}_3^+ + \hat{r}\hat{a}_4^+)^n g_m \frac{1}{\sqrt{m!}}(r\hat{a}_3 + t\hat{a}_4)^m |0\rangle}$$

Example: One-photon input state

$$|\psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^+|0\rangle$$

$$|\psi_{out}\rangle = (t\hat{a}_3^+ + r\hat{a}_4^+)|0\rangle = t|1\rangle_3|0\rangle_4 + r|0\rangle_3|1\rangle_4$$

Example: Classical & Quantum Beamsplitters

Switch to Schrödinger Picture

General input state: 2-mode vacuum

$$\boxed{|\psi_{in}\rangle = \sum_{nm} f_n \frac{1}{\sqrt{n!}} (\hat{a}_1^{+})^n g_m \frac{1}{\sqrt{m!}} (\hat{a}_2^{+})^m |0\rangle}$$

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$$|\psi_{in}\rangle = |1\rangle_1 |0\rangle_2 = \hat{a}_1^{+}|0\rangle$$

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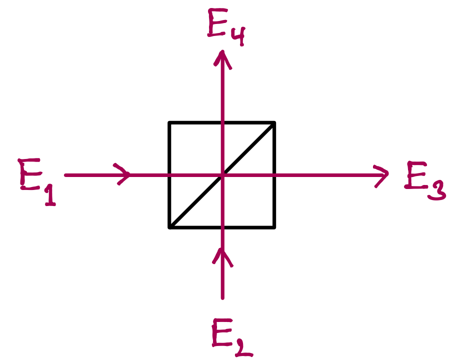
50/50 Beamsplitter



$$t = 1/\sqrt{2},\ r = i/\sqrt{2}$$

$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_3|0\rangle_4 + i|0\rangle_3|1\rangle_4\right)}$$

Note: This is a Mode Entangled State

(*) A coherent superposition of states w/ one photon in port 3 and zero in port 4, and zero in port 3 and one in port 4.

We cannot assign states such as

$\cancel{\frac{1}{\sqrt{2}}(|1\rangle_3 + i|0\rangle_3)}$ to port 3

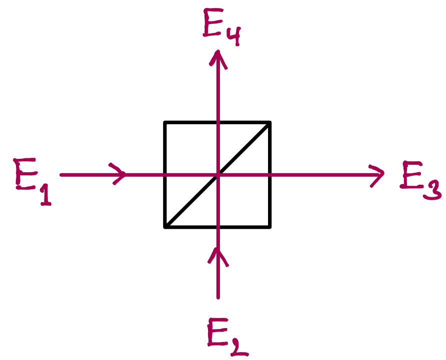
$\cancel{\frac{1}{\sqrt{2}}(|0\rangle_4 + i|1\rangle_4)}$ to port 4

Viewed on their own, each port is in a mixed state

Example: Classical & Quantum Beamsplitters

50/50 Beamsplitter

$t = 1/\sqrt{2},\ r = i/\sqrt{2}$



$$\boxed{|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_3|0\rangle_4 + i\,|0\rangle_3|1\rangle_4\right)}$$

Note: This is a **Mode Entangled State**

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We <u>cannot</u> assign states such as

$\cancel{\frac{1}{\sqrt{2}}\left(|1\rangle_3 + i|0\rangle_3\right)}$ to port 3

$\cancel{\frac{1}{\sqrt{2}}\left(|0\rangle_4 + i|1\rangle_4\right)}$ to port 4

Viewed on their own, each port is in a <u>mixed state</u>

Example: Two-photon input state, 50/50 BS

$$|\psi_{in}\rangle = \hat{a}_1^{+}\hat{a}_2^{+}|0\rangle$$

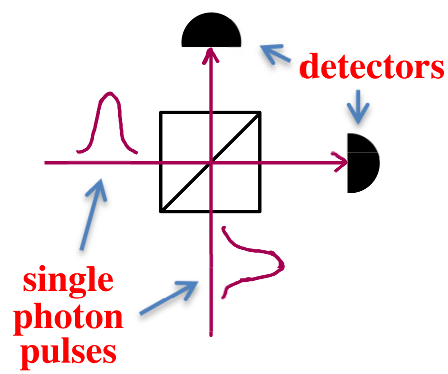
$$|\psi_{out}\rangle = \frac{1}{2}\left(\hat{a}_3^{+} + i\hat{a}_4^{+}\right)\left(i\hat{a}_3^{+} + \hat{a}_4^{+}\right)|0\rangle$$

destructive interference

$$= \frac{1}{2}\left(i\hat{a}_3^{+}\hat{a}_3^{+} + i\hat{a}_4^{+}\hat{a}_4^{+} + \hat{a}_3^{+}\hat{a}_4^{+} - \hat{a}_4^{+}\hat{a}_3^{+}\right)|0\rangle$$

$$= \frac{i}{2}\left(\hat{a}_3^{+}\hat{a}_3^{+} + \hat{a}_4^{+}\hat{a}_4^{+}\right)|0\rangle = \frac{i}{\sqrt{2}}\left(|2\rangle_3|0\rangle_4 + |0\rangle_3|2\rangle_4\right)$$

Experiment:



Coincidence detections are never seen when pulses overlap -> "bunching".

Delay between pulses leads to Coincidence detections.