

Quantum Electrodynamics – QED

Starting point: Maxwells Equations

- (1) $\nabla \cdot \vec{E}(\vec{r}, t) = \frac{1}{\epsilon_0} \rho(\vec{r}, t)$
- (2) $\nabla \cdot \vec{B}(\vec{r}, t) = 0$
- (3) $\nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{r}, t)$
- (4) $\nabla \times \vec{B}(\vec{r}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) + \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{r}, t)$

Implicit: Charges & Fields in Vacuum
No “medium response”

Same issue as with our introductory example:
Maxwells eqs are non-local



We need to put the classical description
in proper form -> Normal Mode expansion

Free Fields - Switch to Fourier Domain

- (1) $i\vec{k} \cdot \vec{E}(\vec{k}, t) = \frac{1}{\epsilon_0} \rho(\vec{k}, t)$
- (2) $i\vec{k} \cdot \vec{B}(\vec{k}, t) = 0$
- (3) $i\vec{k} \times \vec{E}(\vec{k}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{k}, t)$
- (4) $i\vec{k} \times \vec{B}(\vec{k}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{k}, t) + \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{k}, t)$

Fourier Transform:
$$\left\{ \begin{array}{l} \nabla \cdot \vec{G} \Leftrightarrow i\vec{k} \cdot \vec{g} \\ \nabla \times \vec{G} \Leftrightarrow i\vec{k} \times \vec{g} \end{array} \right.$$

Note: This is a Normal Mode decomposition

No charges -> No coupling between modes
with different \vec{k}

Quantum Electrodynamics – QED

Free Fields - Switch to Fourier Domain

$$(1) \quad i\vec{k} \cdot \vec{E}(\vec{k}, t) = \frac{1}{\epsilon_0} \rho(\vec{k}, t)$$

$$(2) \quad i\vec{k} \cdot \vec{B}(\vec{k}, t) = 0$$

$$(3) \quad i\vec{k} \times \vec{E}(\vec{k}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{k}, t)$$

$$(4) \quad i\vec{k} \times \vec{B}(\vec{k}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{k}, t) + \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{k}, t)$$

Fourier Transform:
$$\begin{cases} \nabla \cdot \vec{G} \Leftrightarrow i\vec{k} \cdot \vec{G} \\ \nabla \times \vec{G} \Leftrightarrow i\vec{k} \times \vec{G} \end{cases}$$

Note: This is a Normal Mode decomposition

No charges \rightarrow No coupling between modes with different \vec{k}

Separate into Transverse & Longitudinal Fields

$$\vec{E}(\vec{k}, t) = \vec{E}_{||}(\vec{k}, t) + \vec{E}_{\perp}(\vec{k}, t)$$

$$\vec{B}(\vec{k}, t) = \cancel{\vec{B}_{||}(\vec{k}, t)} + \vec{B}_{\perp}(\vec{k}, t) \quad \text{MEq (2)}$$

↑ Entirely Transverse

Note:
$$\begin{cases} -\frac{i}{k} i\vec{k} \cdot \vec{E} \text{ is the projection of } \vec{E} \text{ onto } \vec{k} \\ \vec{E}_{||} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{E} \text{ onto } \vec{k} \end{cases}$$

MEq (1)

$$\vec{E}_{||} = \frac{\vec{k}}{k} \epsilon_{||} = \frac{\vec{k}}{k} \left(-\frac{i}{k} i\vec{k} \cdot \vec{E} \right) = -i \frac{\vec{k}}{\epsilon_0 k^2} \rho(\vec{k}, t)$$

Coulomb field from the charges

Only \vec{E}_{\perp} and \vec{B}_{\perp} are new degrees of freedom beyond the particles \rightarrow Free Fields

Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

$$\vec{E}(\vec{k}, t) = \vec{E}_{||}(\vec{k}, t) + \vec{E}_{\perp}(\vec{k}, t)$$

$$\vec{B}(\vec{k}, t) = \cancel{\vec{B}_{||}(\vec{k}, t)} + \vec{B}_{\perp}(\vec{k}, t) \quad \text{MEq (2)}$$

↑ Entirely Transverse

Note: $\left\{ \begin{array}{l} -\frac{i}{k} i \vec{k} \cdot \vec{E} \text{ is the projection of } \vec{E} \text{ onto } \vec{k} \\ \vec{E}_{||} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{E} \text{ onto } \vec{k} \end{array} \right.$

MEq (1)

$$\vec{E}_{||} = \frac{\vec{k}}{k} E_{||} = \frac{\vec{k}}{k} \left(-\frac{i}{k} i \vec{k} \cdot \vec{E} \right) = -i \frac{\vec{k}}{\epsilon_0 k^2} \rho(\vec{k}, t)$$

Coulomb field from the charges



Only \vec{E}_{\perp} and \vec{B}_{\perp} are new degrees of freedom beyond the particles -> Free Fields

Eqs for Transverse Fields, from MEqs (3) & (4)

$$(5a) \quad \frac{\partial}{\partial t} \vec{B}(\vec{k}, t) = -i \vec{k} \times \vec{E}_{\perp}(\vec{k}, t)$$

$$(6a) \quad \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{k}, t) = c^2 i \vec{k} \times \vec{B}(\vec{k}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{k}, t)$$

$$(3) \quad i \vec{k} \times \vec{E}(\vec{k}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{k}, t)$$

$$(4) \quad i \vec{k} \times \vec{B}(\vec{k}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{k}, t) + \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{k}, t)$$

Quantum Electrodynamics – QED

Separate into Transverse & Longitudinal Fields

$$\vec{E}(\vec{k}, t) = \vec{E}_{||}(\vec{k}, t) + \vec{E}_{\perp}(\vec{k}, t)$$

$$\vec{B}(\vec{k}, t) = \cancel{\vec{B}_{||}(\vec{k}, t)} + \vec{B}_{\perp}(\vec{k}, t) \quad \text{MEq (2)}$$

↑ Entirely Transverse

Note: $\left\{ \begin{array}{l} -\frac{i}{k} \vec{k} \cdot \vec{E} \text{ is the projection of } \vec{E} \text{ onto } \vec{k} \\ \vec{E}_{||} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{E} \text{ onto } \vec{k} \end{array} \right.$

MEq (1)

$$\vec{E}_{||} = \frac{\vec{k}}{k} E_{||} = \frac{\vec{k}}{k} \left(-\frac{i}{k} \vec{k} \cdot \vec{E} \right) = -i \frac{\vec{k}}{\epsilon_0 k^2} \rho(\vec{k}, t)$$

Coulomb field from the charges

Only \vec{E}_{\perp} and \vec{B}_{\perp} are new degrees of freedom beyond the particles -> Free Fields

Eqs for Transverse Fields, from MEqs (3) & (4)

$$(5a) \quad \frac{\partial}{\partial t} \vec{B}(\vec{k}, t) = -i \vec{k} \times \vec{E}_{\perp}(\vec{k}, t)$$

$$(6a) \quad \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{k}, t) = c^2 i \vec{k} \times \vec{B}(\vec{k}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{k}, t)$$

inverse FT

$$(5b) \quad \frac{\partial}{\partial t} \vec{B}(\vec{r}, t) = -\nabla \times \vec{E}_{\perp}(\vec{r}, t)$$

$$(6b) \quad \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{r}, t) = c^2 \nabla \times \vec{B}(\vec{r}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{r}, t)$$

combine (5b) & (6b)

Wave Equation for the Free Fields

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_{\perp}(\vec{r}, t) = \frac{1}{\epsilon_0 c^2} \frac{\partial}{\partial t^2} \vec{j}_{\perp}(\vec{r}, t)$$

Quantum Electrodynamics – QED

Eqs for Transverse Fields, from MEqs (3) & (4)

$$(5a) \quad \frac{\partial}{\partial t} \vec{B}(\vec{k}, t) = -i\vec{k} \times \vec{E}_{\perp}(\vec{k}, t)$$

$$(6a) \quad \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{k}, t) = c^2 i\vec{k} \times \vec{B}(\vec{k}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{k}, t)$$



inverse FT

$$(5b) \quad \frac{\partial}{\partial t} \vec{B}(\vec{r}, t) = -\nabla \times \vec{E}_{\perp}(\vec{r}, t)$$

$$(6b) \quad \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{r}, t) = c^2 \nabla \times \vec{B}(\vec{r}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{r}, t)$$



combine (5b) & (6b)

Wave Equation for the Free Fields

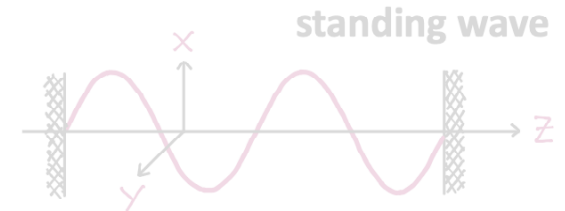
$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_{\perp}(\vec{r}, t) = \frac{1}{\epsilon_0 c^2} \frac{\partial}{\partial t^2} \vec{j}_{\perp}(\vec{r}, t)$$

Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z, t) = \vec{e}_x E_x(z, t)$ and expand

$$(7) \quad E_x(z, t) = \sum_j A_j q_j(t) \sin(k_j z), \quad A_j = \sqrt{\frac{\omega_j^2 m_j}{2 \epsilon_0 V}}$$

fiducial mass



MEq (4) w/no charges

$$\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{r}, t) = \vec{e}_x \frac{1}{c^2} \sum_j A_j \dot{q}_j(t) \sin(k_j z)$$

$$(4) \quad \nabla \times \vec{B}(\vec{r}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) + \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{r}, t)$$

Quantum Electrodynamics – QED

Eqs for Transverse Fields, from MEqs (3) & (4)

$$(5a) \quad \frac{\partial}{\partial t} \vec{B}(\vec{k}, t) = -i\vec{k} \times \vec{E}_{\perp}(\vec{k}, t)$$

$$(6a) \quad \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{k}, t) = c^2 i\vec{k} \times \vec{B}(\vec{k}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{k}, t)$$



inverse FT

$$(5b) \quad \frac{\partial}{\partial t} \vec{B}(\vec{r}, t) = -\nabla \times \vec{E}_{\perp}(\vec{r}, t)$$

$$(6b) \quad \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{r}, t) = c^2 \nabla \times \vec{B}(\vec{r}, t) - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{r}, t)$$



combine (5b) & (6b)

Wave Equation for the Free Fields

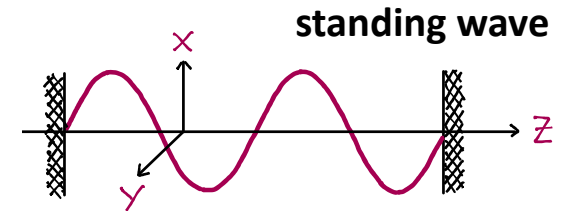
$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_{\perp}(\vec{r}, t) = \frac{1}{\epsilon_0 c^2} \frac{\partial}{\partial t^2} \vec{j}_{\perp}(\vec{r}, t)$$

Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z, t) = \hat{e}_x E_x(z, t)$ and expand

fiducial mass

$$(7) \quad E_x(z, t) = \sum_j A_j q_j(t) \sin(k_j z), \quad A_j = \sqrt{\frac{\omega_j^2 m_j}{2 \epsilon_0 V}}$$

MEq (4) w/no charges

$$\begin{aligned} \nabla \times \vec{B} &= \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}_{\perp}(\vec{r}, t) = \hat{e}_x \frac{1}{c^2} \sum_j A_j \dot{q}_j(t) \sin(k_j z) \\ &= \hat{e}_x \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial z} \right) = -\hat{e}_x \frac{\partial B_y}{\partial z} \end{aligned}$$

\vec{B} transverse $\Rightarrow B_z = 0$

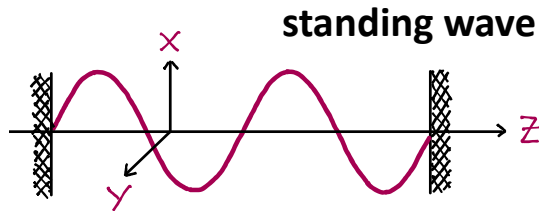
Quantum Electrodynamics – QED

Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{E}_x E_x(z,t)$ and expand

fiducial mass

$$(7) \quad E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \quad A_j = \sqrt{\frac{\omega_j^2 m_j}{2 \epsilon_0 V}}$$

MEq (4) w/no charges

$$\begin{aligned} \nabla \times \vec{B} &= \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}_\perp(\vec{r},t) = \vec{E}_x \frac{1}{c^2} \sum_j A_j \dot{q}_j(t) \sin(k_j z) \\ &= \vec{E}_x \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) = -\vec{E}_x \frac{\partial B_y}{\partial z} \end{aligned}$$

\vec{B} transverse $\Rightarrow B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{E}_z \Rightarrow \vec{B}(z,t) = \vec{E}_y B_y(z,t)$$

$$(5a) \quad \frac{\partial}{\partial t} \vec{B}(\vec{k},t) = -i \vec{k} \times \vec{E}_\perp(\vec{k},t)$$

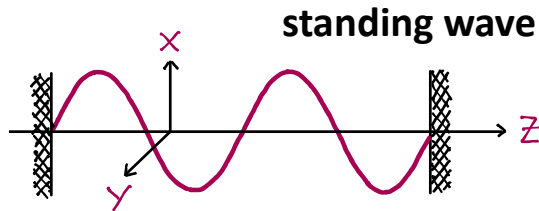
Quantum Electrodynamics – QED

Normal Modes in a 1D Cavity

Length L

Cross section A

Volume $V = LA$



Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{E}_x E_x(z,t)$ and expand

fiducial mass

$$(7) E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \quad A_j = \sqrt{\frac{\omega_j^2 m_j}{2 \epsilon_0 V}}$$

MEq (4) w/no charges

$$\begin{aligned} \nabla \times \vec{B} &= \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}_\perp(z,t) = \vec{E}_x \frac{1}{c^2} \sum_j A_j \dot{q}_j(t) \sin(k_j z) \\ &= \vec{E}_x \left(\frac{\partial B_z}{\partial z} - \frac{\partial B_y}{\partial z} \right) = -\vec{E}_x \frac{\partial B_y}{\partial z} \end{aligned}$$

\vec{B} transverse $\Rightarrow B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{E}_z \Rightarrow \vec{B}(z,t) = \vec{E}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = - \sum_j \frac{A_j}{c^2} \dot{q}_j(t) \sin(k_j z)$$



$$(8) B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

Hamiltonian (Energy) for the Classical Field

$$\begin{aligned} \mathcal{H} &= \frac{\epsilon_0 A}{2} \int_0^L dz (|\vec{E}|^2 + c^2 |\vec{B}|^2) = \\ &= \frac{\epsilon_0 A}{2} \int_0^L dz \sum_j \left[A_j^2 \dot{q}_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j^2} \dot{q}_j(t)^2 \cos^2(k_j z) \right] \end{aligned}$$

Quantum Electrodynamics – QED

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{E}_z \Rightarrow \vec{B}(z,t) = \vec{E}_y B_y(z,t)$$

Putting this together we get

$$\frac{\partial B_y}{\partial z} = - \sum_j \frac{A_j}{c^2} \ddot{q}_j(t) \sin(k_j z)$$



$$(8) \quad B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\epsilon_0 A}{2} \int_0^L dz (|\vec{E}|^2 + c^2 |\vec{B}|^2) =$$

$$\frac{\epsilon_0 A}{2} \int_0^L dz \sum_j \left[A_j^2 \dot{q}_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j^2} \dot{q}_j(t)^2 \cos^2(k_j z) \right]$$

Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{\omega_j^2 m_j}{2\epsilon_0 V}$ we finally get

$$\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]$$

Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\epsilon_0 A}{2} \int_0^L dz (c^2 |\vec{B}|^2 - |\vec{E}|^2) \quad \checkmark$$

$$= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

Check $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{E}_\perp(\vec{r}, t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

Quantum Electrodynamics – QED

Integrating over the Cavity volume

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

and substituting $A_j^2 = \frac{\omega_j^2 m_j}{2\epsilon_0 V}$ we finally get

$$\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]$$

Lagrangian for the Classical Field

$$\begin{aligned} \mathcal{L} &= \frac{\epsilon_0 A}{2} \int_0^L dz (c^2 |\vec{B}|^2 - |\vec{E}|^2) \quad \checkmark \\ &= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right] \end{aligned}$$

Check $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_\perp(\vec{r}_j, t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

And Finally:

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j$$

**As before, a collection
of Harmonic Oscillators,
ready for quantization!**

End 03-31-2021

Today's Jump-off Point

$$E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z)$$

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

$$A_j = \sqrt{\frac{\omega_j^2 m_j}{2 \epsilon_0 V}}$$

$$\mathcal{L} = \frac{\epsilon_0 A}{2} \int_0^L dz (c^2 |\vec{B}|^2 - |\vec{E}|^2)$$

$$= \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j$$

$$\mathcal{H} = \sum_j \left[\frac{1}{2} m_j \dot{q}_j^2 + \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

Classical Fields

Dimensionless Field Variables:

$$Q_j = q_j / q_{0j}, \quad q_{0j} = \sqrt{2\hbar / m_j \omega_j}$$

$$P_j = p_j / p_{0j}, \quad p_{0j} = \sqrt{2\hbar m_j \omega_j}$$



$$\alpha_j(t) = Q_j(t) + i P_j(t) = \alpha_j(0) e^{-i\omega_j t}$$



$$E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \quad \mathcal{E}_j = A_j q_{0j} = \sqrt{\frac{\hbar \omega_j}{\epsilon_0 V}}$$

$$= \sum_j \mathcal{E}_j [\alpha_j(t) + \alpha_j^*(t)] \sin(k_j z)$$

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

$$= -\frac{i}{c} \sum_j \mathcal{E}_j [\alpha_j(t) - \alpha_j^*(t)] \cos(k_j z)$$

↑
field
"per photon"

Quantum Electrodynamics – QED

Classical Fields

Dimensionless Field Variables:

$$Q_j = q_j / q_{0,j}, \quad q_{0,j} = \sqrt{2\hbar / m_j \omega_j}$$

$$P_j = p_j / p_{0,j}, \quad p_{0,j} = \sqrt{2\hbar m_j \omega_j}$$



$$\alpha_j(t) = Q_j(t) + iP_j(t) = \alpha_j(0) e^{-i\omega_j t}$$



$$E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \quad \mathcal{E}_j = A_j q_{0,j} = \sqrt{\frac{\hbar \omega_j}{\epsilon_0 V}}$$

$$= \sum_j \mathcal{E}_j [\alpha_j(t) + \alpha_j^*(t)] \sin(k_j z)$$

↑
field
"per photon"

$$B_y(z,t) = \sum_j \frac{A_j}{k_j c^2} \dot{q}_j(t) \cos(k_j z)$$

$$= -\frac{i}{c} \sum_j \mathcal{E}_j [\alpha_j(t) - \alpha_j^*(t)] \cos(k_j z)$$

Standard Quantization Procedure

$$q_j \rightarrow \hat{q}_j, \quad p_j \rightarrow \hat{p}_j, \quad [\hat{q}_j, \hat{p}_{j'}] = i\hbar \delta_{jj'}$$

$$\alpha_j(t) \rightarrow \hat{a}_j, \quad \alpha_j^*(t) \rightarrow \hat{a}_j^\dagger, \quad [\hat{a}_j, \hat{a}_{j'}^\dagger] = \delta_{jj'}$$

$$\hat{E}_x(z) = \sum_j \mathcal{E}_j (\hat{a}_j + \hat{a}_j^\dagger) \sin(k_j z)$$

$$\hat{B}_y(z) = -\frac{i}{c} \sum_j \mathcal{E}_j (\hat{a}_j - \hat{a}_j^\dagger) \cos(k_j z)$$

Total Field

$$\hat{\vec{E}}(z) = \hat{E}_x \vec{e}_x + \hat{E}_y \vec{e}_y$$

$$\hat{\vec{B}}(z) = \hat{B}_x \vec{e}_x + \hat{B}_y \vec{e}_y$$

Quantum Electrodynamics – QED

Standard Quantization Procedure

$$\begin{aligned}
 q_j &\rightarrow \hat{q}_j & [\hat{q}_j, \hat{p}_j] &= i\hbar \delta_{jj'} \\
 p_j &\rightarrow \hat{p}_j & & \\
 \alpha_j(t) &\rightarrow \hat{a}_j & [\hat{a}_j, \hat{a}_j^\dagger] &= \delta_{jj'} \\
 \alpha_j^*(t) &\rightarrow \hat{a}_j^\dagger & &
 \end{aligned}$$

$$\begin{aligned}
 \hat{E}_x(z) &= \sum_j \epsilon_j (\hat{a}_j + \hat{a}_j^\dagger) \sin(k_j z) \\
 \hat{B}_y(z) &= -\frac{i}{c} \sum_j \epsilon_j (\hat{a}_j - \hat{a}_j^\dagger) \cos(k_j z)
 \end{aligned}$$

Total Field

$$\begin{aligned}
 \hat{\vec{E}}(z) &= \hat{E}_x \hat{x} + \hat{E}_y \hat{y} \\
 \hat{\vec{B}}(z) &= \hat{B}_x \hat{x} + \hat{B}_y \hat{y}
 \end{aligned}$$

Note:

These are the Field Operators in the Schrödinger Picture (**t**-dependence in states)

Often advantageous to use Heisenberg Picture (**t**-dependence in operators)



$$\alpha_j(t) \rightarrow \hat{a}_j(t) = \hat{a}_j(0) e^{-i\omega_j t}$$

Field Quantization in Free Space:

Normal Modes : $\mu_{\vec{k}, \lambda}(\vec{r}) = \vec{\epsilon}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + c.c.$

Finite quantization volume: $\epsilon_{\vec{k}} = \sqrt{\hbar \omega_j / 2 \epsilon_0 V}$

$L \times L \times L$

L large -> nature of boundary conditions not important



Periodic boundary conditions

$|\vec{k}| = n 2\pi / L$

Quantum Electrodynamics – QED

Note:

These are the Field Operators in the Schrödinger Picture (t -dependence in states)

Often advantageous to use Heisenberg Picture (t -dependence in operators)



$$\alpha_j(t) \rightarrow \hat{a}_j(t) = \hat{a}_j(0)e^{-i\omega_j t}$$

Field Quantization in Free Space:

Normal Modes : $u_{\vec{k},\lambda}(\vec{r}) = \vec{\epsilon}_{\vec{k},\lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + c.c.$

Finite quantization volume: $\mathcal{E}_{\vec{k}} = \sqrt{\hbar \omega_{\vec{k}} / 2 \epsilon_0 V}$

$L \times L \times L$

L large \rightarrow nature of boundary conditions not important



Periodic boundary conditions

$|\vec{k}| = n 2\pi / L$

Classical Fields (Fourier Sum):

$$\vec{E}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \mathcal{E}_{\vec{k}, \lambda} \alpha_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + c.c.$$

λ : polarization index

$$\vec{B}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k}, \lambda}}{k c} \mathcal{E}_{\vec{k}, \lambda} \alpha_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + c.c.$$

Quantization:

$$\alpha_{\vec{k}, \lambda} \rightarrow \hat{a}_{\vec{k}, \lambda}, \quad [\hat{a}_{\vec{k}, \lambda}, \hat{a}_{\vec{k}', \lambda'}^\dagger] = \delta_{\vec{k}, \vec{k}'} \delta_{\lambda, \lambda'}$$

$$\alpha_{\vec{k}, \lambda}^* \rightarrow \hat{a}_{\vec{k}, \lambda}^\dagger, \quad [\hat{a}_{\vec{k}, \lambda}, \hat{a}_{\vec{k}', \lambda'}] = [\hat{a}_{\vec{k}, \lambda}^\dagger, \hat{a}_{\vec{k}', \lambda'}^\dagger] = 0$$



$$\hat{\vec{E}}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \mathcal{E}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + H.c.$$


$$\hat{\vec{B}}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k}, \lambda}}{k c} \mathcal{E}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + H.c.$$

– Heisenberg Picture –

Quantum Electrodynamics – QED

Classical Fields (Fourier Sum):

$$\vec{E}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \epsilon_{\vec{k}, \lambda} \alpha_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + c.c.$$


λ: polarization index

$$\vec{B}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k}, \lambda}}{kc} \epsilon_{\vec{k}, \lambda} \alpha_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} + c.c.$$

Quantization:

$$\alpha_{\vec{k}, \lambda} \rightarrow \hat{a}_{\vec{k}, \lambda}, \quad [\hat{a}_{\vec{k}, \lambda}, \hat{a}_{\vec{k}', \lambda'}^\dagger] = \delta_{\vec{k}, \vec{k}'} \delta_{\lambda, \lambda'}$$

$$\alpha_{\vec{k}, \lambda}^* \rightarrow \hat{a}_{\vec{k}, \lambda}^\dagger, \quad [\hat{a}_{\vec{k}, \lambda}, \hat{a}_{\vec{k}', \lambda'}] = [\hat{a}_{\vec{k}, \lambda}^\dagger, \hat{a}_{\vec{k}', \lambda'}^\dagger] = 0$$



$$\hat{\vec{E}}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \epsilon_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + H.c.$$

$$\hat{\vec{B}}_L(\vec{r}, t) = \sum_{\vec{k}, \lambda} \frac{\vec{k} \times \vec{\epsilon}_{\vec{k}, \lambda}}{kc} \epsilon_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})} + H.c.$$

– Heisenberg Picture –

Positive & Negative Frequency Components:

$$\hat{\vec{E}}_L(\vec{r}, t) = \hat{\vec{E}}^{(+)}(\vec{r}, t) + \hat{\vec{E}}^{(-)}(\vec{r}, t)$$

$$\hat{\vec{E}}^{(+)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \epsilon_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})}$$

$$\hat{\vec{E}}^{(-)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda}^* \epsilon_{\vec{k}, \lambda}^* \hat{a}_{\vec{k}, \lambda}^\dagger e^{i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})}$$

Wrap Up:

Read page 13 in handwritten Note Set for brief discussion of different, equivalent ways to put the QED formalism together, e. g.

$$\hat{E}_x \propto [\hat{a}_y + \hat{a}_y^\dagger] \quad \& \quad \hat{B}_y \propto [\hat{a}_z - \hat{a}_z^\dagger]$$

VS

$$\hat{E}_x \propto [\hat{a}_y - \hat{a}_y^\dagger] \quad \& \quad \hat{B}_y \propto [\hat{a}_z + \hat{a}_z^\dagger]$$

Quantum Electrodynamics – QED

Positive & Negative Frequency Components:

$$\hat{\vec{E}}_{\perp}(\vec{r}, t) = \hat{E}^{(+)}(\vec{r}, t) + \hat{E}^{(-)}(\vec{r}, t)$$

$$\hat{E}^{(+)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda} \epsilon_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})}$$

$$\hat{E}^{(-)}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \vec{\epsilon}_{\vec{k}, \lambda}^* \epsilon_{\vec{k}, \lambda}^* \hat{a}_{\vec{k}, \lambda}^{\dagger} e^{-i(\omega_{\vec{k}, \lambda} t - \vec{k} \cdot \vec{r})}$$

Wrap Up:

Read page 13 in handwritten Note Set for brief discussion of different, equivalent ways to put the QED formalism together, e. g.

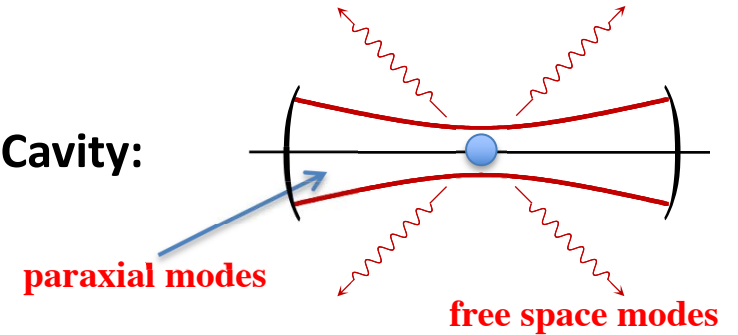
$$\hat{E}_x \propto [\hat{a}_j + \hat{a}_j^{\dagger}] \quad \& \quad \hat{B}_y \propto [\hat{a}_j - \hat{a}_j^{\dagger}]$$

VS

$$\hat{E}_x \propto [\hat{a}_j - \hat{a}_j^{\dagger}] \quad \& \quad \hat{B}_y \propto [\hat{a}_j + \hat{a}_j^{\dagger}]$$

Other Normal Modes Sets

Atom in Cavity:



Wavepackets: (Milloni & Eberly, Sec. 12.8, p 381) (QED lecture notes, p 16)

Classical field

pulse envelope

$$\vec{E}(\vec{r}, t) = \vec{\epsilon} \epsilon_0 \mu(z-ct) e^{i(k_0 z - \omega_0 t)} + c.c.$$

Mode volume

$$V = \int d^3r |\mu(x, y, z-ct)|^2$$

Quantization

$$\epsilon_0 \rightarrow \epsilon_{\vec{k}} \alpha_{\vec{k}} \rightarrow \epsilon_{\vec{k}} \hat{a}_{\vec{k}} \quad \text{etc.}$$

Wave-Particle Duality similar for Photons and Phonons