# Quantum Electrodynamics – QED

## Starting point: Maxwells Equations

(1) $\nabla\cdot\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$

(2) $\nabla\cdot\vec{B}(\vec{r},t) = 0$

(3) $\nabla\times\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$

(4) $\nabla\times\vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\varepsilon_0 c^2}\vec{j}(\vec{r},t)$

**Implicit:** Charges & Fields in Vacuum
No "medium response"

**Same issue as with our introductory example:**

**Maxwells eqs are non-local**

**We need to put the classical description in proper form -> Normal Mode expansion**

## Free Fields - Switch to Fourier Domain

(1) $i\vec{k}\cdot\vec{E}(\vec{k},t) = \frac{1}{\varepsilon_0}\rho(\vec{k},t)$

(2) $i\vec{k}\cdot\vec{B}(\vec{k},t) = 0$

(3) $i\vec{k}\times\vec{E}(\vec{k},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{k},t)$

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**Fourier Transform:**
$$\begin{cases} \nabla\cdot\vec{G} \rightleftarrows i\vec{k}\cdot\vec{g} \\ \nabla\times\vec{G} \rightleftarrows i\vec{k}\times\vec{g} \end{cases}$$

**Note:** This is a Normal Mode decomposition

**No charges -> No coupling between modes with different $\vec{k}$**

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## Separate into Transverse & Longitudinal Fields

$$\vec{\mathcal{E}}(\vec{k},t) = \vec{\mathcal{E}}_{\parallel}(\vec{k},t) + \vec{\mathcal{E}}_{\perp}(\vec{k},t)$$

$$\vec{B}(\vec{k},t) = \cancel{\vec{B}_{\parallel}(\vec{k},t)} + \vec{B}_{\perp}(\vec{k},t) \quad \text{MEq (2)}$$

Entirely Transverse

**Note:** $\begin{cases} -\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}} \text{ is the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \\ \vec{\mathcal{E}}_{\parallel} \text{ is } \frac{\vec{k}}{k} \times \text{ the projection of } \vec{\mathcal{E}} \text{ onto } \vec{k} \end{cases}$

MEq (1)

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = -i\frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

**Coulomb field from the charges**

**Only $\vec{\mathcal{E}}_{\perp}$ and $\vec{B}_{\perp}$ are new degrees of freedom beyond the particles -> <u>Free Fields</u>**

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**Note:**
- $-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}$ **is the projection of** $\vec{\mathcal{E}}$ **onto** $\vec{k}$
- $\vec{\mathcal{E}}_{\parallel}$ is $\frac{\vec{k}}{k}$ × **the projection of** $\vec{\mathcal{E}}$ **onto** $\vec{k}$

**MEq (1)**

$$\vec{\mathcal{E}}_{\parallel} = \frac{\vec{k}}{k}\mathcal{E}_{\parallel} = \frac{\vec{k}}{k}\left(-\frac{i}{k}\, i\vec{k}\cdot\vec{\mathcal{E}}\right) = -i\frac{\vec{k}}{\varepsilon_0 k^2}\rho(\vec{k},t)$$

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## Eqs for Transverse Fields, from MEqs (3) & (4)

**(5a)** $\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k}\times\vec{\mathcal{E}}_{\perp}(\vec{k},t)$

**(6a)** $\frac{\partial}{\partial t}\vec{\mathcal{E}}_{\perp}(\vec{k},t) = c^2 i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) - \frac{1}{\varepsilon_0}\vec{j}_{\perp}(\vec{k},t)$

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**inverse FT**

**(5b)** $\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla\times\vec{E}_{\perp}(\vec{r},t)$

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**combine (5b) & (6b)**

**Wave Equation for the Free Fields**

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_{\perp}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}_{\perp}(\vec{r},t)$$

# Quantum Electrodynamics – QED

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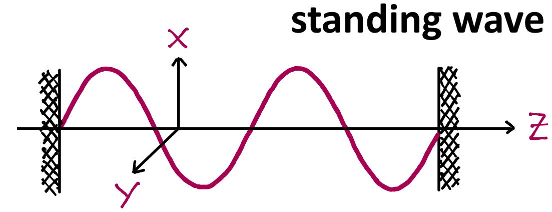
## Normal Modes in a 1D Cavity

Length $L$

Cross section $A$

Volume $V = LA$

standing wave




### Normal Modes are Standing Waves

Let $\vec{E}(z,t) = \vec{\mathcal{E}}_x E_x(z,t)$ and expand

fiducial mass

(7) $$E_x(z,t) = \sum_j A_j q_j(t)\sin(k_j z), \quad A_j = \sqrt{\frac{\omega_j^2 m_j}{2\varepsilon_0 V}}$$

MEq (4) w/no charges

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}_\perp(\vec{r},t) = \vec{\mathcal{E}}_x\frac{1}{c^2}\sum_j A_j\dot{q}_j(t)\sin(k_j t)$$

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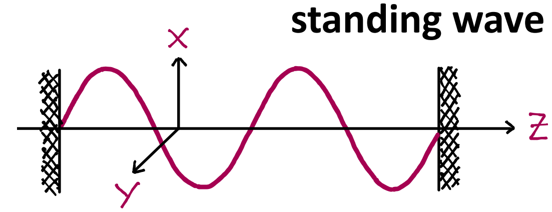


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$$= \vec{\varepsilon}_x\left(\cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z}\right) = -\vec{\varepsilon}_x\frac{\partial B_y}{\partial z}$$

$\mathcal{B}$ transverse ⇨ $B_z = 0$

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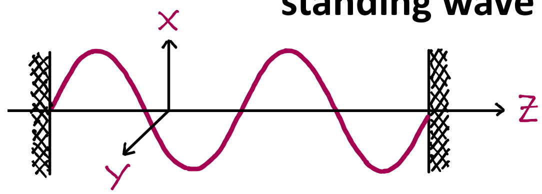


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$\mathcal{B}$ transverse $\Rightarrow$ $B_z = 0$

From Eq. (5a) we see that

$$\vec{B} \perp \vec{E}, \vec{\varepsilon}_z \Rightarrow \vec{B}(z,t) = \vec{\varepsilon}_y B_y(z,t)$$

(5a) $\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t) = -i\vec{k} \times \vec{\mathcal{E}}_\perp(\vec{k},t)$

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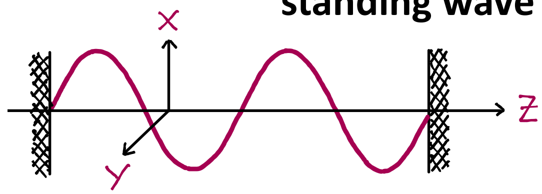


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Putting this together we get

$$\frac{\partial B_y}{\partial z} = -\sum_j \frac{A_j}{c^2} q_j(t)\sin(k_j z)$$

$$\boxed{(8)\quad B_y(z,t) = \sum_j \frac{A_j}{k_j c^2}\dot{q}_j(t)\cos(k_j z)}$$

## Hamiltonian (Energy) for the Classical Field

$$\mathcal{H} = \frac{\varepsilon_0 A}{2}\int_0^L dz\left(|\vec{E}|^2 + c^2|\vec{B}|^2\right) =$$

$$\frac{\varepsilon_0 A}{2}\int_0^L dz \sum_j \left[A_j^2 q_j(t)^2 \sin^2(k_j z) + \frac{A_j^2}{k_j}\dot{q}_j(t)^2\cos^2(k_j z)\right]$$

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$$\Downarrow$$

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**Integrating over the Cavity volume**

$$A \int_0^L dz \sin^2(k_j z) = A \int_0^L dz \cos^2(k_j z) = V/2$$

**and substituting** $A_j^2 = \frac{\omega_j^2 m_j}{2 \varepsilon_0 V}$ **we finally get**

$$\boxed{\mathcal{H} = \sum_j \left[ \frac{1}{2} m_j \omega_j^2 q_j^2 + \frac{1}{2} m_j \dot{q}_j^2 \right]}$$

## Lagrangian for the Classical Field

$$\mathcal{L} = \frac{\varepsilon_0 A}{2} \int_0^L dz \left( c^2 |\vec{B}|^2 - |\vec{E}|^2 \right)$$

$$= \sum_j \left[ \frac{1}{2} m_j \dot{q}_j^2 - \frac{1}{2} m_j \omega_j^2 q_j^2 \right]$$

**Check** $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}_\perp(\vec{r},t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$$

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☑

**Check** $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_\perp(\vec{r},t) = 0 \Rightarrow \ddot{q}_j + \omega_j^2 q_j = 0$

**And Finally:**

$$\boxed{p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m_j \dot{q}_j}$$

## As before, a collection of Harmonic Oscillators, ready for quantization!

End 03-31-2021