OPTI 544: Homework Set #7
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- Keep a copy of your Solution Set -

I

The Lagrangian for a chain of masses m separated by distances a and connected by springs with spring constants κ can be expressed in terms of the particle positions and velocities as

$$\mathscr{L} = \frac{1}{2}\sum_i m\dot{x}_i^2 - \kappa\left(x_{i+1} - x_i\right)^2$$

Starting from this Lagrangian, follow the steps in the notes "Introduction to Field Theory" and derive a wave equation for the displacement field $\eta(x)$ in the continuous limit, $a \to 0$.

II

Consider the x-polarized electromagnetic field in a cylindrical cavity of length L and cross sectional area A. Using the standard normal mode expansion of $E_x(z,t)$ and $B_y(z,t)$ from class, write down expressions for the Hamiltonian and the Lagrangian in terms of the generalized coordinates $q_j(t)$ and their time derivatives.

(a) Use the Lagrange equations of motion to derive a second order differential equation for the $q_j(t)$'s.

(b) Substitute the normal mode expansion of $E_x(z,t)$ in the wave equation and derive a second order differential equation for the $q_j(t)$'s. Compare to the result in (a) above.

III

Consider a *single* standing-wave normal mode of the electromagnetic field in a 1D cavity of length L and cross-sectional area A.

(a) Find the commutator of $\hat{E}_x(z)$ and $\hat{B}_y(z')$ for this mode, and show that in general one cannot simultaneously measure these electric and magnetic fields with arbitrary precision.

(b) Find the minimum uncertainty product $\Delta E_x(z)\Delta B_y(z')$ for this mode.

IV

Consider in the following a 4-port beamsplitter with $t = 1/\sqrt{2}$ and $r = i/\sqrt{2}$.

(a) Let the input state be $|\Psi_{in}\rangle = \left(\sqrt{1-\varepsilon}|1\rangle_1 + \sqrt{\varepsilon}|2\rangle_1\right)\left(\sqrt{1-\varepsilon}|1\rangle_2 + \sqrt{\varepsilon}|2\rangle_2\right)$.

i. e. the wavepackets entering each port are mostly one-photon states but contain a small admixture of two-photon states. Find the output state $|\Psi_{out}\rangle$.

We use photomultiplier type detectors to measure the outputs from the beamsplitter. These detectors will click once when struck by a pulse of one or more photons.

(b) Find the probability of a coincidence detection as function of the two-photon contamination ε.