

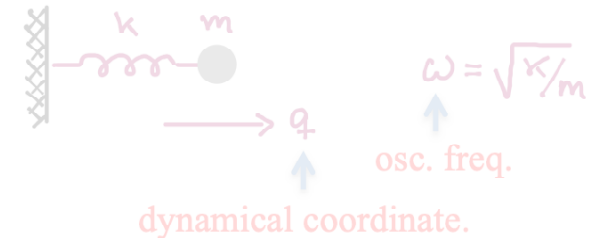
Quantum Electrodynamics – Intro to Field Theory

- (* Primary goal of OPTI 544:
Quantum description of EM field
- (* Challenge: 1st semester Grad level QM (OPTI 570) does not tell how to do this
- (* Warm-up: Quantum field theory for vibrations (sound) in elastic rod
- (* This is in part a review of the classical Lagrange/Hamilton-Jacobi description of continuous systems
- (* Here we present the formalism as a Cookbook Recipe for how we get from Classical to Quantum Physics

See, e. g., Cohen-Tannoudji Vol. 2,
Appendix III, Sections 1-3.

Classical Simple Harmonic Oscillator (SHO)

Particle on
a spring



Kinetic Energy:

$$T = \frac{1}{2} m \dot{q}^2$$

Potential Energy:

$$V = \frac{1}{2} k q^2 = \frac{1}{2} m \omega^2 q^2$$

Lagrangian:

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$$

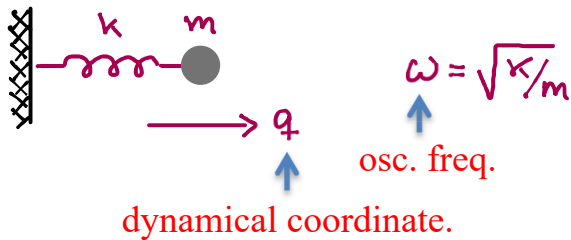
$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

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Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} k q^2 = \frac{1}{2} m \omega^2 q^2$

Lagrangian: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \rightarrow \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

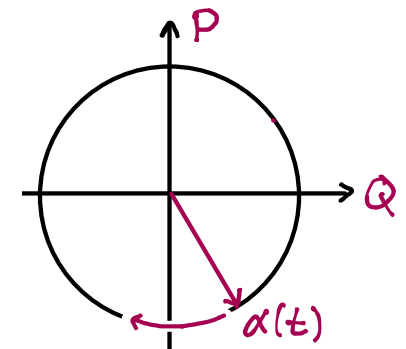
$$\left. \begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -m \omega^2 q \end{aligned} \right\} \rightarrow \ddot{q} + \omega^2 q = 0$$

Scaled variables

$$Q \equiv q/q_0, \quad P = p/p_0$$

$$\alpha = Q + iP \begin{cases} Q = \text{Re}[\alpha] \\ P = \text{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{cases}$$

Phase plane



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Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$$

Hamiltonian

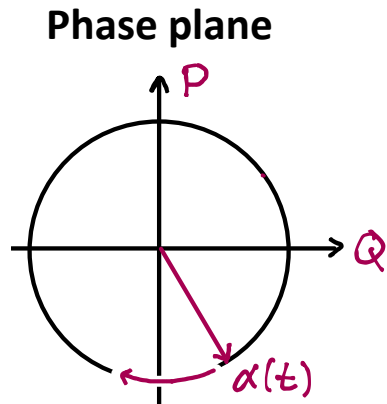
$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

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Scaled variables

$$Q \equiv q/q_0, \quad P = p/p_0$$

$$\alpha = Q + iP \quad \left\{ \begin{array}{l} Q = \text{Re}[\alpha] \\ P = \text{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{array} \right.$$



Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \rightarrow \hat{q}, \quad p \rightarrow \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar$$

Choose $E_0 = \hbar\omega$ \rightarrow $q_0 = \sqrt{\frac{2\hbar}{m\omega}}$, $p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \rightarrow \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{q} + i\frac{\hat{p}}{m\omega} \right)$$

$$[\hat{a}, \hat{a}^\dagger] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega (\hat{Q}^2 + \hat{P}^2) = \hbar\omega (\hat{a}^\dagger \hat{a} + 1/2)$$

$$\hat{N} = \hat{a}^\dagger \hat{a} \quad (\text{number operator})$$

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Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \rightarrow \hat{q}, \quad p \rightarrow \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar$$

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Rewrite:

$$\hat{H} = \hbar\omega (\hat{Q}^2 + \hat{P}^2) = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{1}{2})$$

$$\hat{N} = \hat{a}^\dagger \hat{a} \quad (\text{number operator})$$

Commutator $[\hat{H}, \hat{N}] = 0$

\rightarrow joint energy/number states $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$$

$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\left. \begin{aligned} [\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned} \right\} \rightarrow$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\hat{a}|0\rangle = 0$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

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Commutator $[\hat{H}, \hat{N}] = 0$

→ joint energy/number states $|n\rangle$

$$\begin{aligned} \hat{H}|n\rangle &= \hbar\omega(n+1/2)|n\rangle \\ \hat{N}|n\rangle &= n|n\rangle \end{aligned}$$

Commutators

$$\left. \begin{aligned} [\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned} \right\}$$



$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

Expectation values for \hat{q} and \hat{p} in number states

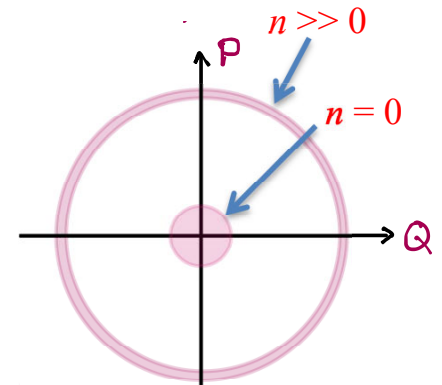
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n+1/2) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n+1/2) \neq 0$$

$$\Delta q \Delta p = \frac{q_0 p_0}{2}(n+1/2) = \hbar(n+1/2)$$

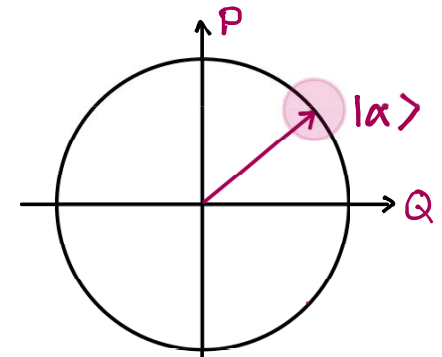
Phase space visualization of number states



Quasi-classical (coherent) state

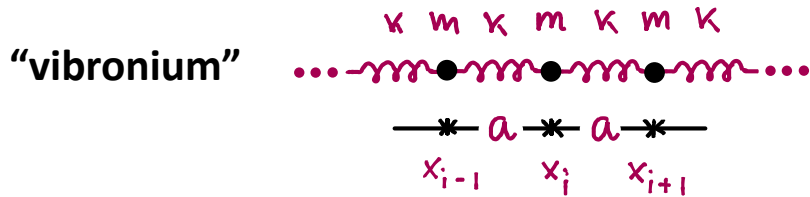
$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_i \frac{\alpha^i}{\sqrt{i!}} |i\rangle$$

$$\Delta q \Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



Quantum Electrodynamics – Intro to Field Theory

Lagrange formulation of 1D Scalar Field



Configuration space = $\{x_i\}$ (set of N osc. positions)

$$T = \sum_{i=1}^N \frac{1}{2} m \dot{x}_i^2, \quad V = \sum_{i=1}^N \frac{1}{2} \kappa (x_{i+1} - x_i)^2$$

Lagrangian, equations of motion

Continuum limit \rightarrow Elastic rod

$$\begin{aligned}
 N \rightarrow \infty & \quad m/a \rightarrow \mu \quad \leftarrow \text{linear mass density} \\
 a \rightarrow dx & \quad \kappa a \rightarrow \gamma \quad \leftarrow \text{Youngs modulus} \\
 \{x_i\} & \rightarrow \eta(x) \quad \leftarrow \text{displacement field (sound)}
 \end{aligned}$$

Rewrite

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \left(\frac{m}{a}\right) \dot{x}_i^2 = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t}\right)^2$$

$$V = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \kappa a \left(\frac{x_{i+1} - x_i}{a}\right)^2 = \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x}\right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t}\right)^2 - \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x}\right)^2$$

Notes, Homework \rightarrow Scalar wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\gamma}{\mu} \frac{\partial^2 \eta}{\partial x^2} = 0$$

– Not yet ready for Quantization –

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Rewrite

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} \left(\frac{m}{a} \dot{x}_i \right)^2 = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2$$

$$V = \lim_{N \rightarrow \infty} \sum_{i=1}^N a \frac{1}{2} k a \left(\frac{x_{i+1} - x_i}{a} \right)^2 = \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x} \right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2 - \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x} \right)^2$$

Notes, Homework \rightarrow Scalar wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\gamma}{\mu} \frac{\partial^2 \eta}{\partial x^2} = 0$$

– Not yet ready for Quantization –

Normal Mode Decomposition

Field in cavity:



Solutions to wave eq.

Let $\eta(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x) \rightarrow$

$$\ddot{\eta} - v^2 \eta'' = -\omega^2 g(t)u(x) - v^2 g(t)u''(x) = 0$$



$$u''(x) = -k^2 u(x), \quad k = \omega/v$$

Solutions in cavity:

$$u_k(x) = \sqrt{\frac{2}{L}} \sin(kx), \quad k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

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Normal Mode Decomposition

Field in cavity:



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Solutions in cavity:

$$u_k(x) = \sqrt{\frac{2}{L}} \sin(kx), \quad k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

These modes are orthonormal and complete

$$\eta(x,t) = \sqrt{L} \sum_k g_k(t) u_k(x)$$

Normal mode expansion of $\eta(x,t)$ in basis $u_k(x)$

Lagrangian for the acoustic field:

$$\begin{aligned} T &= \int dx \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2 = \sum_{k,k'} \frac{1}{2} \underbrace{\mu L}_{M} \dot{g}_k \dot{g}_{k'} \underbrace{\int dx u_k(x) u_{k'}(x)}_{\delta_{kk'}} \\ &= \sum_k \frac{1}{2} M \dot{g}_k^2 \\ V &= \int dx \frac{1}{2} \gamma \left(\frac{\partial \eta}{\partial x} \right)^2 = \sum_{k,k'} \frac{1}{2} \gamma L g_k g_{k'} \int dx \left(\frac{\partial u_k}{\partial x} \right) \left(\frac{\partial u_{k'}}{\partial x} \right) \\ &= \sum_k \frac{1}{2} M \omega_k^2 g_k^2 \end{aligned}$$

$$\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} M \dot{g}_k^2 - \frac{1}{2} M \omega_k^2 g_k^2 \right) = \sum_k \mathcal{L}_k$$

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Jump-off Point for Today's Lecture

$$\mathcal{L} = T - V = \sum_{\mathbf{k}} \left(\frac{1}{2} M \dot{q}_{\mathbf{k}}^2 - \frac{1}{2} M \omega_{\mathbf{k}}^2 q_{\mathbf{k}}^2 \right) = \sum_{\mathbf{k}} \mathcal{L}_{\mathbf{k}}$$



Canonical
Momentum

$$p_{\mathbf{k}} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{\mathbf{k}}} = M \dot{q}_{\mathbf{k}}$$

Hamiltonian

$$\mathcal{H}(\{p_{\mathbf{k}}, q_{\mathbf{k}}\}) = T + V = \sum_{\mathbf{k}} \left(\frac{p_{\mathbf{k}}^2}{2M} + \frac{1}{2} M \omega_{\mathbf{k}}^2 q_{\mathbf{k}}^2 \right)$$

(collection of SHO's, one for each normal mode)

Following the standard recipe...

$$E_{0,\mathbf{k}} = \hbar \omega_{\mathbf{k}}, \quad q_{0,\mathbf{k}} = \sqrt{2\hbar/M\omega_{\mathbf{k}}}, \quad p_{0,\mathbf{k}} = \sqrt{2M\hbar\omega_{\mathbf{k}}}$$

$$Q_{\mathbf{k}} = q_{\mathbf{k}}/q_{0,\mathbf{k}}, \quad P_{\mathbf{k}} = p_{\mathbf{k}}/p_{0,\mathbf{k}}, \quad \alpha_{\mathbf{k}} = Q_{\mathbf{k}} + iP_{\mathbf{k}}$$

... we get solutions

$$\alpha_{\mathbf{k}}(t) = Q_{\mathbf{k}}(t) + iP_{\mathbf{k}}(t) = \alpha_{\mathbf{k}}(0) e^{-i\omega_{\mathbf{k}}t}$$

This finally gives us

$$\mathcal{H} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} (Q_{\mathbf{k}}^2 + P_{\mathbf{k}}^2) = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \alpha_{\mathbf{k}}^* \alpha_{\mathbf{k}}$$

$$y(x,t) = \sqrt{L} \sum_{\mathbf{k}} q_{\mathbf{k}}(t) u_{\mathbf{k}}(x)$$

$$= \frac{i}{2} \sum_{\mathbf{k}} \sqrt{L} q_{0,\mathbf{k}} \left(\alpha_{\mathbf{k}}(t) u_{\mathbf{k}}(x) + \alpha_{\mathbf{k}}^*(t) u_{\mathbf{k}}^*(x) \right)$$

allows real or complex $u_{\mathbf{k}}(x)$

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$$\alpha_{\mathbf{k}}(t) = Q_{\mathbf{k}}(t) + iP_{\mathbf{k}}(t) = \alpha_{\mathbf{k}}(0) e^{-i\omega_{\mathbf{k}}t}$$

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$$y(x, t) = \sqrt{L} \sum_{\mathbf{k}} q_{\mathbf{k}}(t) u_{\mathbf{k}}(x)$$

$$= \frac{1}{2} \sum_{\mathbf{k}} \sqrt{L q_{0,\mathbf{k}}^2} \left(\alpha_{\mathbf{k}}(t) u_{\mathbf{k}}(x) + \alpha_{\mathbf{k}}^*(t) u_{\mathbf{k}}^*(x) \right)$$

allows real or complex $u_{\mathbf{k}}(x)$

Formal Quantization Procedure:

$$q_{\mathbf{k}} \rightarrow \hat{q}_{\mathbf{k}}, \quad p_{\mathbf{k}} \rightarrow \hat{p}_{\mathbf{k}}, \quad \alpha_{\mathbf{k}} \rightarrow \hat{a}_{\mathbf{k}}$$

$$[\hat{q}_{\mathbf{k}}, \hat{p}_{\mathbf{k}'}] = i\hbar \delta_{\mathbf{k}\mathbf{k}'}, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k}\mathbf{k}'}, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0$$

Note: $\mathbf{k} \neq \mathbf{k}' \Rightarrow$ operators commute
(normal modes = independent degs. of freedom)

Hamiltonian & Quantized fields

$$\hat{H} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} (\hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}} + 1/2)$$

$$\hat{\eta}(x) = \sqrt{L} \sum_{\mathbf{k}} \hat{q}_{\mathbf{k}} u_{\mathbf{k}}(x) = \sum_{\mathbf{k}} \sqrt{L q_{0,\mathbf{k}}^2} (\hat{a}_{\mathbf{k}} u_{\mathbf{k}}(x) + \hat{a}_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^*(x))$$

$$\hat{\pi}(x) = \frac{1}{\sqrt{L}} \sum_{\mathbf{k}} \hat{p}_{\mathbf{k}} u_{\mathbf{k}}(x) = -i \sum_{\mathbf{k}} \sqrt{\frac{\hbar q_{0,\mathbf{k}}^2}{L}} (\hat{a}_{\mathbf{k}} u_{\mathbf{k}}(x) - \hat{a}_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^*(x))$$

field $\hat{\eta}(x)$ and canonical momentum field $\hat{\pi}(x)$

$$\Rightarrow [\hat{\eta}(x), \hat{\pi}(x')] = i\hbar \delta(x-x')$$

Quantum Electrodynamics – Intro to Field Theory

Formal Quantization Procedure:

$$q_k \rightarrow \hat{q}_k, \quad p_k \rightarrow \hat{p}_k, \quad a_k \rightarrow \hat{a}_k$$

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}] = 0$$

Note: $k \neq k' \Rightarrow$ operators commute
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Hamiltonian & Quantized fields

$$\hat{H} = \sum_k \hbar \omega_k (\hat{a}_k^\dagger \hat{a}_k + 1/2)$$

$$\hat{\eta}(x) = \sqrt{L} \sum_k \hat{q}_k u_k(x) = \sum_k \sqrt{L \frac{\hbar \omega_k}{2}} (\hat{a}_k u_k(x) + \hat{a}_k^\dagger u_k^*(x))$$

$$\hat{\Pi}(x) = \frac{1}{\sqrt{L}} \sum_k \hat{p}_k u_k(x) = -i \sum_k \sqrt{\frac{\hbar \omega_k}{L}} (\hat{a}_k u_k(x) - \hat{a}_k^\dagger u_k^*(x))$$

field $\hat{\eta}(x)$ and canonical momentum field $\hat{\Pi}(x)$

$$\Rightarrow [\hat{\eta}(x), \hat{\Pi}(x')] = i\hbar \delta(x-x')$$

Quantum States:

The quantum field is a collection of QSHO's



State space = tensor product of SHO spaces

Fock Space $\mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \dots, \mathcal{E}_{k_j}$ SHO space

Fock State $|\{n_{k_1}, n_{k_2}, \dots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots \otimes |n_{k_j}\rangle$ SHO state

$\hat{a}_{k_i}, \hat{a}_{k_i}^\dagger$ destroy/create excitations in mode k_i

Vacuum State $|0\rangle$ zero quanta in every mode

Favorite Question: What is a Phonon?

Quantum Electrodynamics – Intro to Field Theory

Quantum States:

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SHO space

Fock State

$$|\{n_{k_1}, n_{k_2}, \dots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots \otimes |n_{k_j}\rangle$$

SHO state

$$\hat{a}_{k_i}, \hat{a}_{k_i}^\dagger \quad \text{destroy/create excitations in mode } k_i$$

Vacuum State

$$|0\rangle \quad \leftarrow \text{zero quanta in every mode}$$

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0 | \hat{\eta}(x) | 0 \rangle =$$

$$\sum_k \frac{1}{2} \sqrt{L g_{0,k}^2} \left(\langle 0 | \hat{a}_k | 0 \rangle u_k(x) + \langle 0 | \hat{a}_k^\dagger | 0 \rangle u_k^*(x) \right) = 0$$

Zero Point Fluctuations

$$\langle 0 | \hat{\eta}(x)^2 | 0 \rangle$$

$$= \sum_k \frac{1}{2} L g_{0,k} g_{0,k} \langle 0 | \hat{a}_k^\dagger \hat{a}_k + 1 | 0 \rangle u_k(x) u_k^*(x)$$

$$= \sum_k \frac{1}{2} L g_{0,k} |u_k(x)|^2 \neq 0$$

$$\sum_{kk'} \langle 0 | (\hat{a}_k u_k + \hat{a}_k^\dagger u_k^*) (\hat{a}_{k'} u_{k'} + \hat{a}_{k'}^\dagger u_{k'}^*) | 0 \rangle =$$

$$\sum_k \langle 0 | (\hat{a}_k \hat{a}_k u_k u_k + \hat{a}_k \hat{a}_k^\dagger u_k u_k^* + \hat{a}_k^\dagger \hat{a}_k u_k^* u_k + \hat{a}_k^\dagger \hat{a}_k^\dagger u_k^* u_k^*) | 0 \rangle =$$

$$\sum_k \langle 0 | \hat{a}_k \hat{a}_k^\dagger u_k u_k^* | 0 \rangle = \sum_k \langle 0 | (\hat{a}_k^\dagger \hat{a}_k + 1) | 0 \rangle u_k u_k^*$$

$$\hat{\eta}(x) = \sum_k \sqrt{L g_{0,k}^2} (\hat{a}_k u_k(x) + \hat{a}_k^\dagger u_k^*(x))$$

Favorite Question: What is a Phonon?

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SHO space

Fock State

$$|\{n_{k_1}, n_{k_2}, \dots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \dots \otimes |n_{k_j}\rangle$$

SHO state

$$\hat{a}_{k_i}, \hat{a}_{k_i}^\dagger \quad \text{destroy/create excitations in mode } k_i$$

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$$|0\rangle \quad \leftarrow \quad \text{zero quanta in every mode}$$

Favorite Question: **What is a Phonon?**

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0 | \hat{\eta}(x) | 0 \rangle =$$

$$\sum_k \frac{1}{2} \sqrt{L g_{0,k}^2} \left(\langle 0 | \hat{a}_k | 0 \rangle \mu_k(x) + \langle 0 | \hat{a}_k^\dagger | 0 \rangle \mu_k^*(x) \right) = 0$$

Zero Point Fluctuations

$$\langle 0 | \hat{\eta}(x)^2 | 0 \rangle$$

$$= \sum_k \frac{1}{2} L g_{0,k} g_{0,k} \langle 0 | \hat{a}_k^\dagger \hat{a}_k + 1 | 0 \rangle \mu_k(x) \mu_k^*(x)$$

$$= \sum_k \frac{1}{2} L g_{0,k} |\mu_k(x)|^2 \neq 0$$

Thus $\Delta \eta(x) \neq 0$ with zero phonons in field

Note the famous divergence:

$$E_{vac} = \langle 0 | \hat{H} | 0 \rangle = \sum_k \frac{\hbar \omega_k}{2} \rightarrow \infty \text{ for } k \rightarrow \infty$$

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Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0 | \hat{\eta}(x) | 0 \rangle =$$

$$\sum_{\mathbf{k}} \frac{1}{2} \sqrt{L g_{0,\mathbf{k}}} (\langle 0 | \hat{a}_{\mathbf{k}} | 0 \rangle u_{\mathbf{k}}(x) + \langle 0 | \hat{a}_{\mathbf{k}}^{\dagger} | 0 \rangle u_{\mathbf{k}}^*(x)) = 0$$

Zero Point Fluctuations

$$\langle 0 | \hat{\eta}(x)^2 | 0 \rangle =$$

$$= \sum_{\mathbf{k}} \frac{1}{2} L g_{0,\mathbf{k}} g_{0,\mathbf{k}} \langle 0 | \hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}} + 1 | 0 \rangle u_{\mathbf{k}}(x) u_{\mathbf{k}}^*(x)$$

$$= \sum_{\mathbf{k}} \frac{1}{2} L g_{0,\mathbf{k}} |u_{\mathbf{k}}(x)|^2 \neq 0$$

Thus $\Delta \eta(x) \neq 0$ with zero phonons in field

Note the famous divergence:

$$E_{\text{vac}} = \langle 0 | \hat{H} | 0 \rangle = \sum_{\mathbf{k}} \frac{\hbar \omega_{\mathbf{k}}}{2} \rightarrow \infty \text{ for } \mathbf{k} \rightarrow \infty$$

Are our Phonons waves or particles?

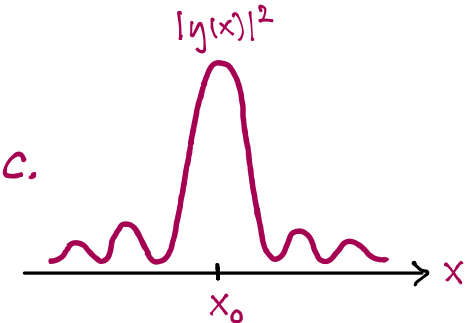
Extended

Localized

Particle-like Phonons

Classical Wavepacket

$$\eta(x) = \sum_{\mathbf{k}} f_{\mathbf{k}} u_{\mathbf{k}}(x) + \text{c.c.}$$



Define $\hat{A}^{\dagger} = \sum_{\mathbf{k}} f_{\mathbf{k}} \hat{a}_{\mathbf{k}}^{\dagger}$, $\sum_{\mathbf{k}} |f_{\mathbf{k}}|^2 = 1$

$$\hat{A}^{\dagger} | 0 \rangle = f_{\mathbf{k}_1} | 1_{\mathbf{k}_1}, 0_{\mathbf{k}_2}, \dots \rangle + f_{\mathbf{k}_2} | 0_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, \dots \rangle + \dots$$

Localized excitation in the field.

These Particle-like Phonons are Bosons

$$\hat{A}^{\dagger} \hat{A}^{\dagger} | 0 \rangle = \hat{A}^{\dagger} \hat{A}^{\dagger} | 0 \rangle$$

1st particle @ x
2nd particle @ x'

1st particle @ x'
2nd particle @ x