Quantum Electrodynamics – Intro to Field Theory

(*) **Primary goal of OPTI 544: Quantum description of EM field**

(*) **Challenge: 1st semester Grad level QM (OPTI 570) does not tell how to do this**

(*) **Warm-up: Quantum field theory for vibrations (sound) in elastic rod**

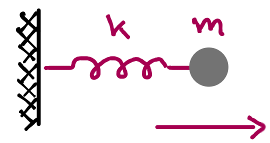
(*) **This is in part a review of the <u>classical</u> Lagrange/Hamilton-Jacobi description of continuous systems**

(*) **Here we present the formalism as a Cookbook Recipe for how we get from Classical to Quantum Physics**

See, e. g., Cohen-Tannoudji Vol. 2, Appendix III, Sections 1-3.

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



$\omega = \sqrt{\kappa/m}$ ← osc. freq.

q ← dynamical coordinate.

Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} \kappa q^2 = \frac{1}{2} m \omega^2 q^2$

<u>Lagrangian</u>: $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

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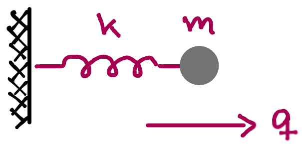

dynamical coordinate.

$\omega = \sqrt{k/m}$

osc. freq.

Kinetic Energy:	$T = \frac{1}{2} m \dot{q}^2$
Potential Energy:	$V = \frac{1}{2} k q^2 = \frac{1}{2} m \omega^2 q^2$

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usual eq. of motion

Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q}$$

Hamiltonian

$$\mathcal{H} = T(\dot{q} = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\left.\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \end{aligned}\right\} \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

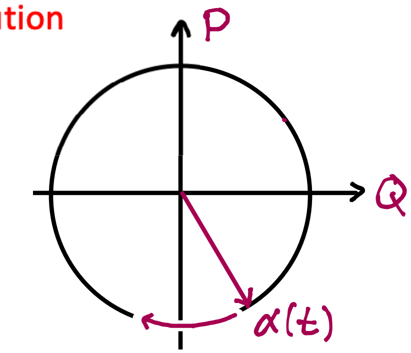
solution

Scaled variables

$Q = q/q_0, \quad P = p/p_0$

$$\alpha = Q + iP \left\{ \begin{aligned} Q &= \mathrm{Re}[\alpha] \\ P &= \mathrm{Im}[\alpha] \\ \mathcal{H} &= E_0 \alpha^* \alpha \end{aligned} \right.$$

Phase plane



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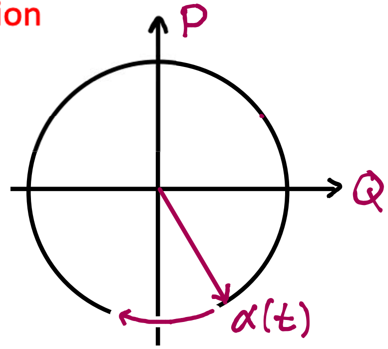

Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$\boxed{q \to \hat{q}, \quad p \to \hat{p}, \quad [\hat{q}, \hat{p}] = i\hbar}$$

Choose $E_0 = \hbar\omega$ $\Rightarrow$ $q_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \to \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right)$$

$$[\hat{a}, \hat{a}^+] = 1$$

Rewrite:

$$\hat{H} = \hbar\omega(\hat{Q}^2 + \hat{P}^2) = \hbar\omega(\hat{a}^+\hat{a} + 1/2)$$

$$\hat{N} = \hat{a}^+\hat{a}$$ **(number operator)**

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Quantum Harmonic Oscillator

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$\hat{N} = \hat{a}^+\hat{a}$ (number operator)

Commutator $[\hat{H}, \hat{N}] = 0$

⇒ **joint energy/number states** $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

$$\hat{N}|n\rangle = n|n\rangle$$

Commutators

$$\left.\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned}\right\} \Rightarrow$$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}|0\rangle = 0$$

Generating excited states

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle$$

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Commutator $[\hat{H}, \hat{N}] = 0$

➡ **joint energy/number states** $|n\rangle$

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Commutators

$$\left.\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}}$$

Generating excited states

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle}$$

Expectation values for $\hat{q}$ and $\hat{p}$ in number states

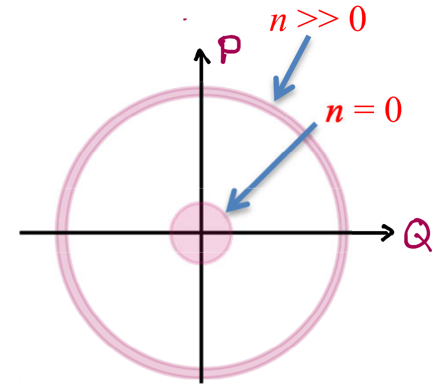
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n+1/2) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n+1/2) \neq 0$$

$$\Delta q\,\Delta p = \frac{q_0 p_0}{2}(n+1/2) = \hbar(n+1/2)$$

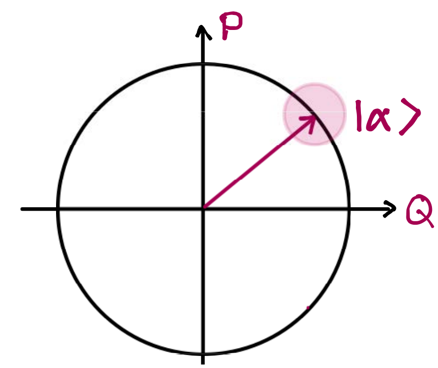
Phase space visualization of number states



Quasi-classical (coherent) state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_i \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\Delta q\,\Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



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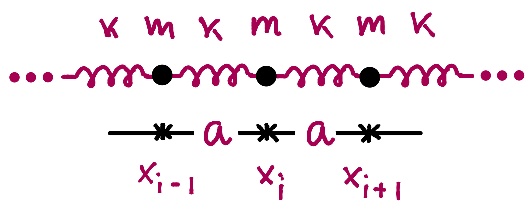
Lagrange formulation of 1D Scalar Field

"vibronium"



Configuration space = $\{x_i\}$ (set of *N* osc. positions)

$$T=\sum_{i=1} \tfrac{1}{2} m \dot{x}_i^2, \quad V=\sum_{i=1}^{N} \tfrac{1}{2}\kappa (x_{i+1}-x_i)^2$$

Lagrangian, equations of motion

Continuum limit ➔ Elastic rod

$N \to \infty$ $\quad m/a \to \mu$ ← linear mass density

$a \to dx$ $\quad \kappa a \to Y$ ← Youngs modulus

$\{x_i\} \to y(x)$ ← displacement field (sound)

Rewrite

$$\boxed{\begin{aligned} T &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^2 = \int dx\, \tfrac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 \\ V &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\kappa a \left(\frac{x_{i+1}-x_i}{a}\right)^2 = \int dx\, \tfrac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2 \end{aligned}}$$

Lagrangian:

$$\boxed{\mathcal{L} = T - V = \int dx\, \tfrac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 - \int dx\, \tfrac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2}$$

Notes, Homework ➔ Scalar wave equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} - \frac{Y}{\mu}\frac{\partial^2 y}{\partial x^2} = 0}$$

– Not yet ready for Quantization –

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Rewrite

$$T = \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^2 = \int dx \tfrac{1}{2}\mu\left(\frac{\partial \eta}{\partial t}\right)^2$$

$$V = \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2}\kappa a\left(\frac{x_{i+1}-x_i}{a}\right)^2 = \int dx \tfrac{1}{2} Y\left(\frac{\partial \eta}{\partial x}\right)^2$$

Lagrangian:

$$\mathcal{L} = T - V = \int dx \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \int dx \tfrac{1}{2} Y\left(\frac{\partial y}{\partial x}\right)^2$$

Notes, Homework ➡ **Scalar wave equation**

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{Y}{\mu}\frac{\partial y^2}{\partial x^2} = 0$$

– Not yet ready for Quantization –

Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$ ➡

$$\ddot{y} - v^2 y'' = -\omega^2 g(t)u(x) - v^2 g(t)u''(x) = 0$$

$$u''(x) = -k^2 u(x), \quad k = \omega/v$$

Solutions in cavity:

$$u_k(x) = \sqrt{\frac{2}{L}}\sin(kx), \quad k = \frac{n\pi}{L}$$

These standing waves are a set of Normal Modes

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Normal Mode Decomposition

Field in cavity:

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These standing waves are a set of Normal Modes

These modes are orthonormal and complete

$$\boxed{y(x,t) = \sqrt{L}\sum_k q_k(t)\, u_k(x)}$$

Normal mode expansion of $y(x,t)$ **in basis** $u_k(x)$

Lagrangian for the acoustic field:

$$T = \int dx \, \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \sum_{k,k'} \frac{1}{2}\underbrace{\mu L}_{M}\, \dot{q}_k \dot{q}_{k'} \underbrace{\int dx\, u_k(x) u_{k'}(x)}_{\delta_{kk'}}$$

$$= \sum_k \frac{1}{2} M \dot{q}_k^2$$

$$V = \int dx \, \frac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2 = \sum_{k,k'} \frac{1}{2} Y L\, q_k q_{k'} \int dx \left(\frac{\partial u_k}{\partial x}\right)\left(\frac{\partial u_{k'}}{\partial x}\right)$$

$$= \sum_k \frac{1}{2} M \omega_k^2 q_k^2$$

$$\boxed{\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 - \frac{1}{2} M \omega_k^2 q_k^2\right) = \sum_k \mathcal{L}_k}$$