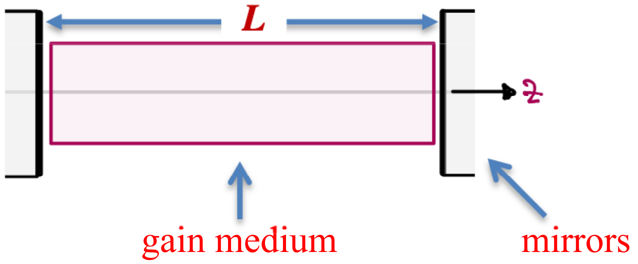
Begin 03-17-2021

Semi-Classical Laser Theory

Lasing action: requires a gain medium & feedback



(∗) As usual we simplify to focus on the key concepts → 1D cavity

(∗) For spherical mirror resonators, see M&E Ch. 14

Optical Resonator/Cavity →

Eigenmodes of the Electromagnetic Field

Plane Parallel Mirrors → standing waves

Length L → wave number for *m*'th mode

$$k = \frac{m\pi}{L}, \quad m \text{ integer}$$

Field in the m'th mode

$$\boxed{\vec{E}_m(z,t) = \vec{\varepsilon}_m \mathcal{E}_m(t) \sin(k_m z) e^{-i\omega t}}$$

Slowly Varying Envelope

Note: Gain → dispersion in cavity

$$\omega \neq k_m c = \omega_m$$

Laser freq.

Vacuum mode freq.

Polarization density, m'th mode

$$\vec{P}_m(z,t) = \vec{\varepsilon}_m 2N\mu^* \rho_{21}^{(m)}(z,t) \sin(k_m z) e^{-i\omega t}$$

$$\vec{P}(z,t) = \sum_m \vec{P}_m(z,t) \longleftarrow$$

Total polarization density in all modes

Note: Saturation effects → Mode cross-talk

Semi-Classical Laser Theory

Wave eq. in a Resonator

– mimic loss by including current $\vec{J} = \sigma \vec{E}$

finite conductivity

4th Maxwell Eq.: $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \sigma \vec{E}$

Wave Eq. in resonator, with distributed loss

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\kappa}{c^2}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\vec{P}$$

$\kappa = \sigma/\varepsilon_0$ ← Phenomenological loss constant (losses + output coupling)

units 1/s

Wave Eq. for $m'th$ mode in the resonator

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\kappa}{c^2}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{\mathcal{E}}_m \mathcal{E}_m(t)\sin(k_m z)e^{-i\omega t}$$

$$= \frac{\vec{\mathcal{E}}_m}{\varepsilon_0 c^2} 2N\mu^* \sin(k_m z)\frac{\partial^2}{\partial t^2}\left(\rho_{21}^{(m)}(t)e^{-i\omega t}\right)$$

Apply SVEA & resonant approx., $\omega - \omega_m \ll \omega$ (HW)

$$\left[-i(\omega-\omega_m) + \frac{\kappa}{2} + \frac{\partial}{\partial t}\right]\mathcal{E}_m(t) = \frac{i\omega}{\varepsilon_0} N\mu^* \rho_{21}^{(m)}(t)$$

Quasi-steady state solution:

Fast atomic response

High-Q cavity

$(\beta \gg \kappa)$

→ $\rho_{21}^{(m)}(t)$ **in S. S. given** $\mathcal{E}(t)$

We can adiabatically eliminate $\rho_{21}^{(m)}(t)$

by replacing w/ S.S value given $\mathcal{E}(t)$, ρ_{11} & ρ_{22}

$$\rho_{21}(t) = -\frac{i\mu\mathcal{E}(t)}{2\hbar}\frac{\beta - i\Delta}{\beta^2+\Delta^2}(\rho_{22}-\rho_{11})$$

Note: We will allow for some external process that potentially creates a population inversion

Semi-Classical Laser Theory

Substitute in Equation for $\mathcal{E}_m(t)$

$$\left[-i(\omega-\omega_m)+\frac{\kappa}{2}+\frac{\partial}{\partial t}\right]\mathcal{E}_m(t)$$

$$=\frac{N|\mu|^2\omega}{2\varepsilon_0\hbar}\frac{\beta-i\Delta}{\beta^2+\Delta^2}(\rho_{22}-\rho_{11})\mathcal{E}_m(t)$$

Let $N_1=N\rho_{11}$, $N_2=N\rho_{22}$ and define

$$g\equiv\frac{|\mu|^2\omega}{\varepsilon_0\hbar c}\frac{\beta}{\Delta^2+\beta^2}(N_2-N_1)=\sigma(\Delta)(N_2-N_1)$$

gain

$$\delta\equiv\frac{\Delta}{\beta}g=\frac{\Delta}{\beta}\sigma(\Delta)(N_2-N_1)$$

dispersion

Fundamental Eq. of Semiclassical Laser Theory

$$\boxed{\frac{\partial}{\partial t}\mathcal{E}_m(t)=\frac{1}{2}\left[-\kappa+2i(\omega-\omega_m)+c(g-i\delta)\right]\mathcal{E}_m(t)}$$

The FESLT gives us insight into

- (∗) **Threshold behavior**
- (∗) **Laser intensity and power output**
- (∗) **Laser frequency and linewidth**

Equation for Laser intensity $I\propto\mathcal{E}^*\mathcal{E}$

$$\frac{dI}{dt}\propto\frac{\partial\mathcal{E}_m^*(t)}{dt}\mathcal{E}_m(t)+C.C.$$

$$=\frac{1}{2}\left[-\kappa-2i(\omega-\omega_m)+c(g-i\delta)\right]|\mathcal{E}_m(t)|^2+C.C.$$

$$\frac{dI}{dt}=(cg-\kappa)I$$

- $g>g_t$: exponential growth
- $g<g_t$: exponential decay

We define $g_t=\sigma(\Delta)\Delta N_t$, $\Delta N_t=\frac{\kappa}{c\sigma(\Delta)}$

Semi-Classical Laser Theory

The FESLT gives us insight into

- (✱) Threshold behavior
- (✱) Laser intensity and power output
- (✱) Laser frequency and linewidth

Equation for Laser intensity $I \propto \mathcal{E}^* \mathcal{E}$

$$\frac{dI}{dt} \propto \frac{\partial \mathcal{E}_m^*(t)}{dt} \mathcal{E}_m(t) + C.C.$$

$$= \frac{1}{2}\left[-\kappa - 2i(\omega - \omega_m) + c(g - i\delta)\right] |\mathcal{E}_m(t)|^2 + C.C.$$

$$\frac{dI}{dt} = (cg - \kappa) I \quad \begin{cases} g > g_t : \text{exponential growth} \\ g < g_t : \text{exponential decay} \end{cases}$$

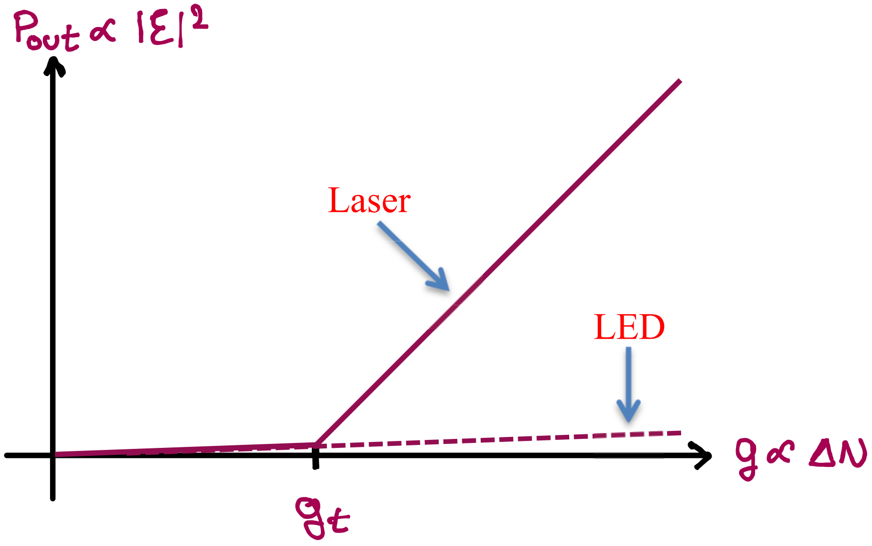
We define $g_t = \sigma(\Delta) \Delta N_t \, , \quad \Delta N_t = \frac{\kappa}{c\sigma(\Delta)}$

These are Key parameters that characterizes a laser

$g_t = \frac{\kappa}{c}$ Threshold Gain

$\Delta N_t = \frac{\kappa}{c\sigma(\Delta)}$ Threshold Inversion

Example: Diode lasers & threshold behavior



End 03-17-2021

Semi-Classical Laser Theory

Begin 03-19-2021

Fundamental Eq. of Semiclassical Laser Theory

$$\boxed{\begin{array}{l}\dfrac{\partial \mathcal{E}_m(t)}{\partial t} = \dfrac{1}{2}\left[-\kappa + 2i(\omega-\omega_m) + c(g - i\delta)\right]\mathcal{E}_m(t) \\[2ex] g \equiv \sigma(\Delta)(N_2-N_1) \qquad \delta \equiv \dfrac{\Delta}{\beta}g = \dfrac{\Delta}{\beta}\sigma(\Delta)(N_2-N_1)\end{array}}$$

gain

dispersion

Laser Frequency in Steady State

Let $\frac{\partial}{\partial t}\mathcal{E}_m = 0$ and consider imaginary part of FESLT

$$i(\omega-\omega_m) - i\frac{c\delta}{2} = 0\,, \quad \text{with } \delta = \frac{\Delta}{\beta}g$$

$$\omega_m - \omega = -\frac{gc}{2\beta}\Delta = \frac{gc}{2\beta}(\omega-\omega_{21})$$

Solve for ω :

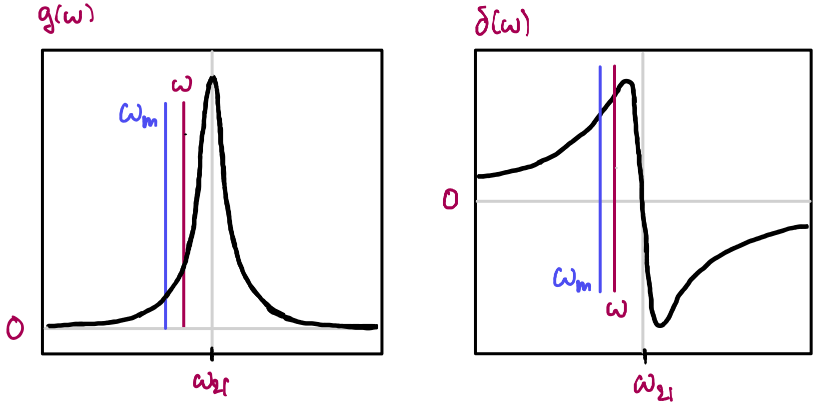
$$\omega = \frac{\omega_m + {}^{gc}/_{2\beta}\,\omega_{21}}{1 + gc/2\beta} \approx \omega_m + \frac{cg}{2\beta}(\omega_{21}-\omega_m)$$

laser frequency

for $\frac{gc}{2\beta} \ll 1$

frequency pulling

Physical interpretation – note $\delta(\omega) > 0$ for $\omega < \omega_{21}$



From MBE's $n_R = 1 - \frac{\delta\omega}{2k}$ ⇨ $n_R < 1$

⇨ Optical L < physical L ⇨ ω increases

– Laser frequency is pulled towards resonance

Semi-Classical Laser Theory

Solve for ω :

$$\omega = \frac{\omega_m + {}^{gc}/_{2\beta}\,\omega_{21}}{1 + gc/2\beta} \approx \omega_m + \frac{cg}{2\beta}(\omega_{21} - \omega_m)$$

laser frequency

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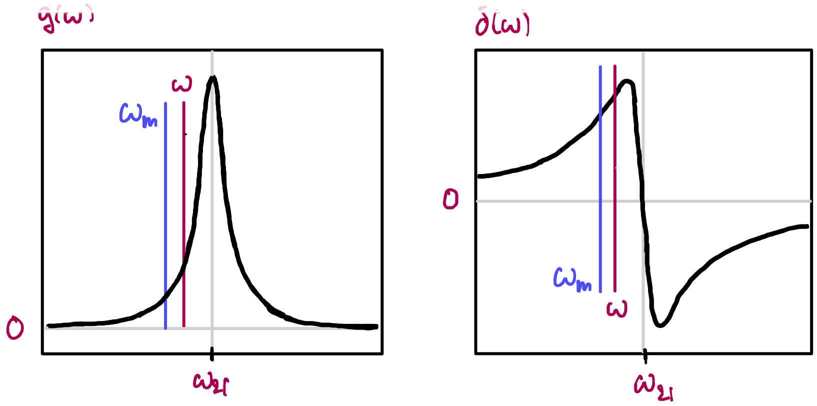


From MBE's $n_R = 1 - \frac{\delta\omega}{2k}$ ⇒ $n_R < 1$

⇒ Optical L < physical L ⇒ ω increases

– Laser frequency is pulled towards resonance

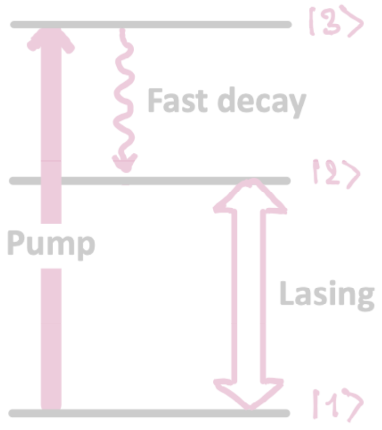
Gain requires Population Inversion ⇒

Laser Pumping Schemes

3-Level System

Ruby Laser

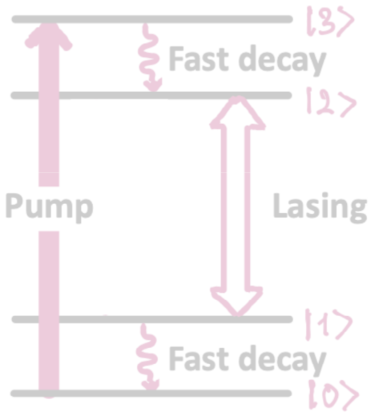
Hard to Pump!



4-Level System

Nd-YAG
Ti-Sappire
Er-Fiber (glass)
Organic Dye
Helium-Neon
Semiconductor

Easy to Pump!



Semi-Classical Laser Theory

Gain requires Population Inversion ⇨

Laser Pumping Schemes

3-Level System

Ruby Laser

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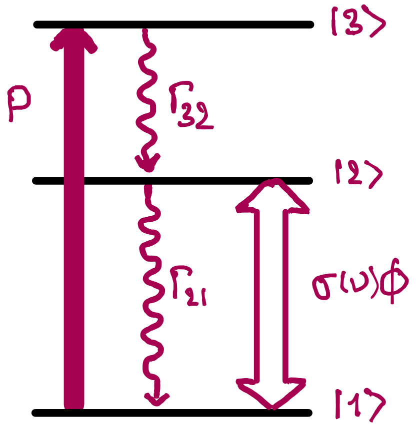
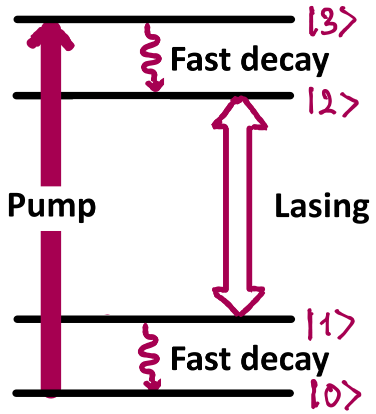


4-Level System

Nd-YAG
Ti-Sappire
Er-Fiber (glass)
Organic Dye
Helium-Neon
Semiconductor

Easy to Pump!



Population Rate Equations – 3 level System

Let $\Gamma_{32} \gg P,\ \Gamma_{21},\ \sigma(\nu)\phi \Rightarrow N_3 \sim 0$

⇩

$$\boxed{\begin{aligned} \dot{N}_1 &= -PN_1 + \Gamma_{21}N_2 + \sigma(\nu)\phi\,(N_2 - N_1) \\ \dot{N}_2 &= PN_1 - \Gamma_{21}N_2 - \sigma(\nu)\phi\,(N_2 - N_1) \end{aligned}}$$

End 03-19-2021

Begin 03-22-2021

Steady State Solution (Home Work Set 6)

$$N_2 - N_1 = \frac{(P - \Gamma_{21})(N_2 + N_1)}{P + \Gamma_{21} + 2\sigma(\nu)\phi}$$

Use $\left\{ \begin{aligned} &N_1 + N_2 = N \\ &g(\nu) = \sigma(\nu)(N_2 - N_1) \end{aligned} \right.$ ⇨

$$g(\nu) = \sigma(\nu)\frac{(P - \Gamma_{21})N}{P + \Gamma_{21} + 2\sigma(\nu)\phi} > 0 \quad \text{iff} \quad P > \Gamma_{21}$$

Semi-Classical Laser Theory

Population Rate Equations – 3 level System

Let $\Gamma_{32} \gg P,\ \Gamma_{21},\ \sigma(\nu)\phi$ ➡ $N_3 \sim 0$

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Steady State Solution (Home Work Set 6)

$$N_2 - N_1 = \frac{(P - \Gamma_{21})(N_2 + N_1)}{P + \Gamma_{21} + 2\sigma(\nu)\phi}$$

Use $\begin{cases} N_1 + N_2 = N \\ g(\nu) = \sigma(\nu)(N_2 - N_1) \end{cases}$ ➡

$$g(\nu) = \sigma(\nu)\frac{(P - \Gamma_{21})N}{P + \Gamma_{21} + 2\sigma(\nu)\phi} > 0 \quad \text{iff} \quad P > \Gamma_{21}$$

Divide top & bottom w/ $P + \Gamma_{21}$

$$g(\nu) = \frac{g_0(\nu)}{1 + \phi/\phi_{sat}}$$

Saturated Gain

$$g_0(\nu) = \sigma(\nu)\frac{(P - \Gamma_{21})N}{P + \Gamma_{21}}$$

Small Signal Gain

$$\phi_{sat} = \frac{P + \Gamma_{21}}{2\sigma(\nu)}$$

Saturation Flux

$$I_{sat} = h\nu\phi_{sat}$$

Saturation Intensity

Semi-Classical Laser Theory

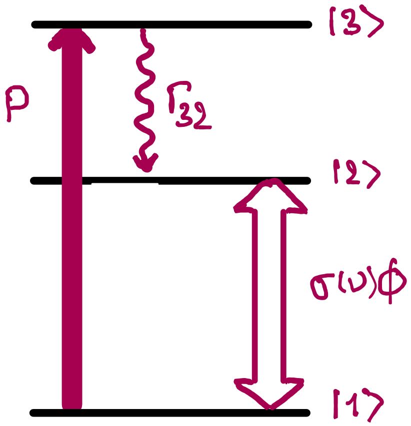
Gain requires Population Inversion ⇨

Laser Pumping Schemes

3-Level System

Ruby Laser

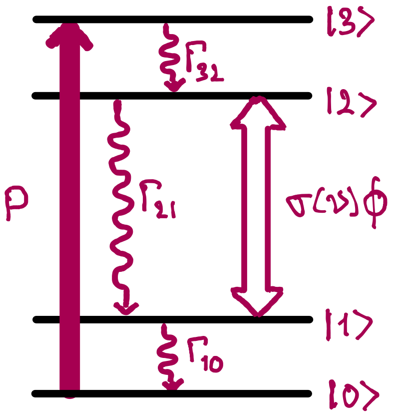
Hard to Pump!



4-Level System

Nd-YAG
Ti-Sappire
Er-Fiber (glass)
Organic Dye
Helium-Neon
Semiconductor

Easy to Pump!



Population Rate Equations – 4 level System

Let $\Gamma_{32} \gg P,\ \Gamma_{21},\ \sigma(\nu)\phi$ ⇨ $N_3 \sim 0$

⇩

$$\dot{N}_0 = -PN_0 + \Gamma_{10}N_1$$

$$\dot{N}_1 = -\Gamma_{10}N_1 + \Gamma_{21}N_2 + \sigma(\nu)\phi(N_2 - N_1)$$

$$\dot{N}_2 = PN_0 - \Gamma_{21}N_2 - \sigma(\nu)\phi(N_2 - N_1)$$

Steady State Solution (Home Work Set 6)

$$N_2 - N_1 = \frac{P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21} + (2P + \Gamma_{10})\sigma\phi}$$

Semi-Classical Laser Theory

Population Rate Equations – 4 level System

Let $\Gamma_{32} \gg P,\ \Gamma_{21},\ \sigma(\nu)\phi$ ➡ $N_3 \sim 0$

$$\dot{N}_0 = -PN_0 + \Gamma_{10} N_1$$

$$\dot{N}_1 = -\Gamma_{10} N_1 + \Gamma_{21} N_2 + \sigma(\nu)\phi(N_2 - N_1)$$

$$\dot{N}_2 = PN_0 - \Gamma_{21} N_2 - \sigma(\nu)\phi(N_2 - N_1)$$

Steady State Solution (Home Work Set 6)

$$N_2 - N_1 = \frac{P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21} + (2P + \Gamma_{10})\sigma\phi}$$

Divide top & bottom w/ $P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}$

$$g(\nu) = \frac{g_0(\nu)}{1 + \phi/\phi_{sat}}$$

Saturated Gain

$$g_0(\nu) = \frac{\sigma(\nu)P(\Gamma_{10} - \Gamma_{21})N}{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}$$

Small Signal Gain

$$\phi_{sat} = \frac{P(\Gamma_{10} + \Gamma_{21}) + \Gamma_{10}\Gamma_{21}}{2(P + \Gamma_{10})\sigma(\nu)}, \quad I_{sat} = h\nu\phi_{sat}$$

Saturation Flux

Saturation Intensity

Semi-Classical Laser Theory

Divide top & bottom w/ $P(\Gamma_{10}+\Gamma_{21})+\Gamma_{10}\Gamma_{21}$

$$g(\nu) = \frac{g_0(\nu)}{1+\phi/\phi_{sat}}$$

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$$g_0(\nu) = \frac{\sigma(\nu)P(\Gamma_{10}-\Gamma_{21})N}{P(\Gamma_{10}+\Gamma_{21})+\Gamma_{10}\Gamma_{21}}$$

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Saturation Flux

Saturation Intensity

Threshold Inversion and Pumping Rates

Example: 3-level system: $\Delta N = \frac{(P-\Gamma_{21})N}{P+\Gamma_{21}+2\sigma\phi}$

By definition $g_t = \sigma(\nu)\Delta N_t$ ➡

Threshold Gain

Threshold Inversion

defines P_t

$$\Delta N_t = \frac{(P_t-\Gamma_{21})N}{P_t+\Gamma_{21}+\underbrace{2\sigma\phi}_{=0}} = \frac{(P_t-\Gamma_{21})N}{P_t+\Gamma_{21}}$$

below threshold

Solve for the Threshold Pumping Rate.

$$P_t^{3\text{-Level}} = \frac{N+\Delta N_t}{N-\Delta N_t}$$

$$P_t^{4\text{-Level}} = \frac{\Delta N_t}{N-\Delta N_t} \quad \text{for } \Gamma_{10} \gg \Gamma_{21}$$

Semi-Classical Laser Theory

Threshold Inversion and Pumping Rates

Example: 3-level system: $\Delta N = \frac{(P-\Gamma_{21})N}{P+\Gamma_{21}+2\sigma\phi}$

By definition $g_t = \sigma(\nu)\Delta N_t$ ⇒

Threshold Gain

Threshold Inversion

defines P_t

$$\Delta N_t = \frac{(P_t-\Gamma_{21})N}{P_t+\Gamma_{21}+\underbrace{2\sigma\phi}_{=0}} = \frac{(P_t-\Gamma_{21})N}{P_t+\Gamma_{21}}$$

below threshold

Solve for the Threshold Pumping Rate.

⇓

$$\boxed{\begin{array}{l} P_t^{3\text{-Level}} = \frac{N+\Delta N_t}{N-\Delta N_t} \\ \\ P_t^{4\text{-Level}} = \frac{\Delta N_t}{N-\Delta N_t} \quad \text{for} \quad \Gamma_{10} \gg \Gamma_{21} \end{array}}$$

Gain under Lasing Conditions

Below threshold $g \leq g_t$ ⇒ $\begin{cases} \phi \leq \phi_{sat} \\ g(\nu) \sim g_0(\nu) \end{cases}$

Small Signal Gain

Above threshold: **exp. growth of** ϕ **until the gain saturates, growth slows and stops**

⇓

Steady State:

$$\boxed{g(\nu) = g_t = \kappa/c}$$

Saturated Gain = Loss

Important Question:

- **What if many modes see significant gain?**
- **It depends, and can be complicated !**

Semi-Classical Laser Theory

Gain under Lasing Conditions

Below threshold $g \leq g_t$ ⇨ $\begin{cases} \phi \leq \phi_{sat} \\ g(\nu) \sim g_0(\nu) \end{cases}$

↑ Small Signal Gain

Above threshold: exp. growth of ϕ until the gain saturates, growth slows and stops

⇩

Steady State:

$$\boxed{g(\nu) = g_t = \kappa/c}$$

Saturated Gain = Loss

Important Question:

– What if many modes see significant gain?

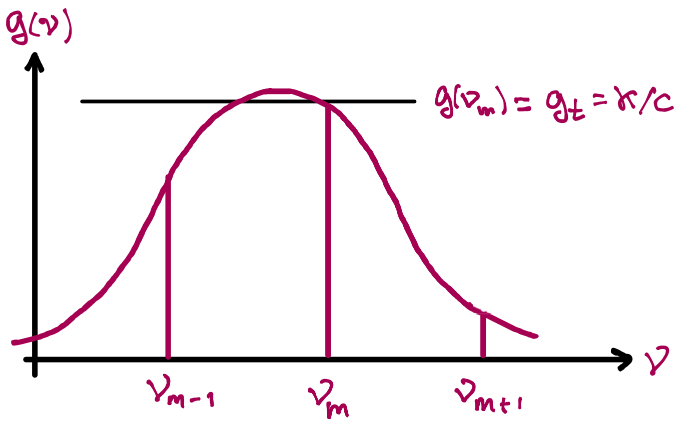
– It depends, and can be complicated !

Homogeneous Gain Broadening

All atoms identical, couple identically to modes (lifetime, collision broadening)

Consider a gradual increase in the pumping rate

$P \sim 0$ ⇨ $P \sim P_t$ ⇨ $P \gg P_t$



1st mode to reach threshold will lase, saturate gain, and clamp the inversion at its threshold value

Mode Competition

⇨ **Winner Takes All**