Emission and Absorption – Population Rate Equations

Blackbody Radiation

This is a standard problem in Optical Physics, which we review only briefly in OPTI 544. See almost any textbook for details, including Milloni & Eberly.

Step (1): We define a Lineshape Function:

$$\sigma(\Delta) = \frac{\pi \mu_{12}^2 \omega_{21}}{2\varepsilon_0 \hbar c} \underbrace{\frac{\delta\nu_{21}/\pi}{(\nu_{21}-\nu)^2 + \delta\nu_{21}^2}}_{S(\nu)}, \quad \delta\nu_{21} = \beta/2\pi$$

$S(\nu)$ is sharply peaked compared to the energy density $g(\nu)$ of Blackbody Radiation.

➡ we can approximate $S(\nu) = \delta(\nu_{21}-\nu)$

Step (2): Sub in the expression for transition rates and integrate over all frequencies

$$R_{12} = \int_0^\infty \sigma(\nu)\phi(\nu)\,d\nu = \frac{\pi \mu_{21}^2 \omega_{21}}{2\varepsilon_0 \hbar} \int_0^\infty S(\nu) \frac{g(\nu)}{h\nu}\,d\nu$$

$$\Rightarrow R_{12} = \boxed{\frac{\pi \mu_{12}^2}{2\varepsilon_0 \hbar^2}} g(\nu_{21}) = B\, g(\nu_{21})$$

← Einstein B coefficient

Step (3): 2-level atom in thermal equilibrium with Blackbody Radiation field

$$\frac{g_{22}}{g_{11}} = e^{-h\nu_{21}/k_B T}$$

$$(A_{21} + R_{21})\, g_{22} = R_{12} g_{11}$$

(Detailed Balance)

$$\Rightarrow \boxed{g(\nu_{21}) = \frac{A_{21}/B}{e^{h\nu_{21}/k_B T} - 1}}$$

Step (4): Use prior result for A_{21} to find A_{21}/B, which is independent of μ_{12}^2

$$\boxed{g(\nu) = \frac{8\pi h\nu^3/c^3}{e^{h\nu/k_B T} - 1}}$$

Energy Density of the Blackbody Radiation field

Step (5): We can extend this to find the

- Total electromagnetic energy density
- Total radiated intensity from a black body

Emission and Absorption – Population Rate Equations

Step (3): 2-level atom in thermal equilibrium with Blackbody Radiation field

$$\frac{g_2}{g_1} = e^{-h\nu_{21}/k_B T}$$

$$(A_{21} + R_{21})\, g_2 = R_{12}$$

(Detailed Balance)

$$\Rightarrow \quad \rho(\nu_{21}) = \frac{A_{21}/B}{e^{h\nu_{21}/k_B T} - 1}$$

Step (4): Use prior result for A_{21} to find A_{21}/B, which is independent of η_{12}^2

$$\rho(\nu) = \frac{8\pi h\nu^3/c^3}{e^{h\nu/k_B T} - 1}$$

Energy Density of the Blackbody Radiation field

Step (5): We can extend this to find the

* Total electromagnetic energy density
* Total radiated intensity from a black body

Step (6): Relative importance of spontaneous and stimulated emission

We have $\dfrac{A_{21}}{R_{12}} = \dfrac{A_{21}}{B\rho(\nu)} = e^{h\nu/k_B T} - 1 \gg 1$

for $T <$ several $\times$ 1000K

Take the surface of the Sun, $T = 5800K$

$$\frac{A_{21}}{R_{12}} \sim 140 \;@\; \lambda_{max} = 500\,nm$$

Vector Model of the 2-Level Atom

We return to the Density Matrix Equations of Motion for a 2-Level atom

$$\dot{\rho}_{11} = -\Gamma_1 \rho_{11} + A_{21}\rho_{22} - \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{22} = -\Gamma_2 \rho_{22} - A_{21}\rho_{22} + \frac{i}{2}(\chi\rho_{12} - \chi^*\rho_{21})$$

$$\dot{\rho}_{12} = (i\Delta - \beta)\rho_{12} + \frac{i\chi^*}{2}(\rho_{22} - \rho_{11}) = \dot{\rho}_{21}^*$$

$$\beta = \frac{1}{\tau} + \frac{1}{2}(\Gamma_1 + \Gamma_2 + A_{21}), \quad \chi = \vec{\mu}_{21}\cdot\vec{\mathcal{E}}_0/\hbar$$

Note: In our previous iteration we studied the Rate Equation approximation, which is useful when we are looking for steady state solutions

Here our goal is different – we seek to recast the Density Matrix formalism in a way that is better suited to understanding and modeling coherent evolution and transient phenomena. This will also be useful when we study wave and light pulse propagation.

Optical Bloch Equations (OBE's)

Let $\Gamma_1 = \Gamma_2 = 0$ ➡ $\rho_{11} + \rho_{22} = 1$, $\rho_{12} = \rho_{21}^*$

➡ **3** independent, real-valued variables

Define Bloch Variables

$$u = \rho_{21} + \rho_{12}$$

$$v = i(\rho_{21} - \rho_{12})$$

$$w = \rho_{22} - \rho_{11}$$

Let $\chi = |\chi| e^{-i\varphi}$, substitute equations for ρ

⬇

Optical Bloch Equations

$$\dot{u} = -\Delta v - |\chi| \sin\varphi\, w$$

$$\dot{v} = \Delta u + |\chi| \cos\varphi\, w$$

$$\dot{w} = -|\chi|(\cos\varphi\, v - \sin\varphi\, u)$$

Vector Model of the 2-Level Atom

Optical Bloch Equations (OBE's)

Let $\Gamma_1 = \Gamma_2 = 0$ ⇒ $\varrho_{11} + \varrho_{22} = 1$, $\varrho_{12} = \varrho_{21}^*$

⇒ **3 independent, real-valued variables**

Define	$u = \varrho_{21} + \varrho_{12}$
Bloch Variables	$v = i(\varrho_{21} - \varrho_{12})$
	$w = \varrho_{22} - \varrho_{11}$

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Optical Bloch Equations

$$\dot{u} = -\Delta v - |\chi| \sin\varphi\, w$$

$$\dot{v} = \Delta u + |\chi| \cos\varphi\, w$$

$$\dot{w} = -|\chi|(\cos\varphi\, v - \sin\varphi\, u)$$

Define: $\vec{S} = u\vec{i} + v\vec{j} + w\vec{k}$

$\vec{Q} = -|\chi|\cos\varphi\,\vec{i} - |\chi|\sin\varphi\,\vec{j} + \Delta\vec{k}$

⇓

$$\dot{\vec{S}} = \vec{Q} \times \vec{S}$$

Torque — Bloch Vector

Equivalent to OBE's

length conserved

Note: $\frac{d}{dt}(S^2) = 2\vec{S}\cdot\dot{\vec{S}} = 2\vec{S}\cdot(\vec{Q}\times\vec{S}) = 0$

From the definition of the Bloch Variables we get

$$\varrho = \frac{1}{2}\begin{pmatrix} 1-w & u+iv \\ u-iv & 1+w \end{pmatrix}$$

$$\mathrm{Tr}\,\varrho^2 = \frac{1}{2}\left[1 + u^2 + v^2 + w^2\right] = \frac{1}{2}\left[1 + |S|^2\right] \le 1$$

⇒ $|S|^2 \le 1$

Vector Model of the 2-Level Atom

Define: $\vec{S} = u\vec{i} + v\vec{j} + w\vec{k}$

$\vec{Q} = -|X|\cos\varphi\,\vec{i} - |X|\sin\varphi\,\vec{j} + \Delta\vec{k}$

$\dot{\vec{S}} = \vec{Q} \times \vec{S}$

Torque — Bloch Vector

Equivalent to OBE's

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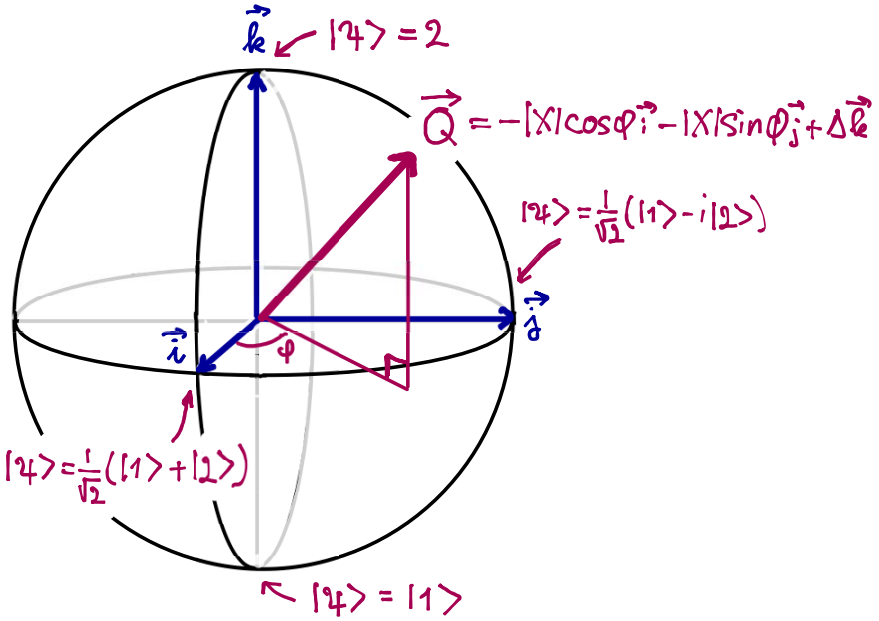
$$\mathrm{Tr}\,\rho^2 = \frac{1}{2}\left[1 + u^2 + v^2 + w^2\right] = \frac{1}{2}\left[1 + |S|^2\right] \le 1$$

$$\Rightarrow \boxed{|S|^2 \le 1}$$

Clearly, $|S^2| = 1 \Rightarrow \mathrm{Tr}\,\rho^2 = 1 \Rightarrow$ **pure state**

States w/ $|\vec{S}| < 1 \Rightarrow \mathrm{Tr}\,\rho^2 < 1 \Rightarrow$ **mixed state**

$$|\vec{S}| = 0 \Rightarrow \mathrm{Tr}\,\rho^2 = 1/2 \Rightarrow \rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

maximally mixed

Note: **The above suggests a physical state can be represented by a vector $\vec{S}$, whose tip lies on the surface of (pure) or inside (mixed) a sphere of unit radius, and whose length is conserved under Schrödinger evolution. This is the Bloch Sphere.**