Introduction to Quantum Field Theory

(*) **Primary goal of OPTI 544:**
Quantum description of EM field

(*) **Challenge: 1st semester Grad level QM (OPTI 570) does not tell how to do this**

(*) **Warm-up: Quantum field theory for vibrations (sound) in elastic rod**

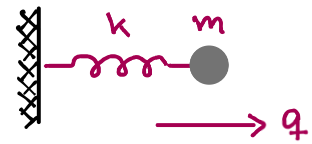
(*) **This is in part a review of the <u>classical</u> Lagrange/Hamilton-Jacobi description of continuous systems**

(*) **Here we present the formalism as a Cookbook Recipe for how we get from Classical to Quantum Physics**

See, e. g., Cohen-Tannoudji Vol. 2, Appendix III, Sections 1-3.

Classical Simple Harmonic Oscillator (SHO)

Particle on a spring



$\omega = \sqrt{\kappa/m}$

osc. freq.

dynamical coordinate.

Kinetic Energy: $T = \frac{1}{2} m \dot{q}^2$

Potential Energy: $V = \frac{1}{2} \kappa q^2 = \frac{1}{2} m \omega^2 q^2$

<u>**Lagrangian:**</u> $\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad \ddot{q} + \omega^2 q = 0$$

usual eq. of motion

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Conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot q} = m\dot q$$

Hamiltonian

$$\mathcal{H} = T(\dot q = p/m) + V(q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

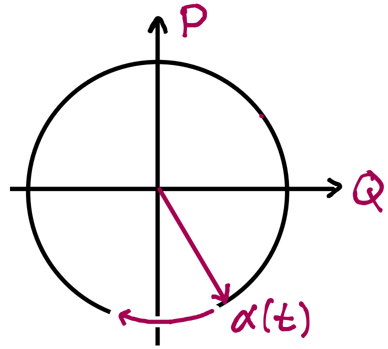
$$\left.\begin{aligned} \dot q &= \frac{\partial \mathcal{H}}{\partial p} = p/m \\ \dot p &= -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \end{aligned}\right\} \Rightarrow \ddot q + \omega^2 q = 0$$

Scaled variables

$$Q \equiv q/q_0, \quad P = p/p_0$$

$$\alpha = Q + iP \begin{cases} Q = \mathrm{Re}[\alpha] \\ P = \mathrm{Im}[\alpha] \\ \mathcal{H} = E_0 \alpha^* \alpha \end{cases}$$

Phase plane



Quantum Harmonic Oscillator

Formal Quantization Procedure:

$$q \to \hat q, \quad p \to \hat p, \quad [\hat q, \hat p] = i\hbar$$

Choose $E_0 = \hbar\omega \Rightarrow q_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2m\hbar\omega}$

natural scale

$$\alpha \to \hat a = \hat Q + i\hat P = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat q + i\frac{\hat p}{m\omega}\right)$$

$$[\hat a, \hat a^+] = 1$$

Rewrite:

$$\hat H = \hbar\omega(\hat Q^2 + \hat P^2) = \hbar\omega(\hat a^+\hat a + 1/2)$$

$\hat N = \hat a^+ \hat a$ (number operator)

Introduction to Quantum Field Theory

Commutator $[\hat{H}, \hat{N}] = 0$

➡ **joint energy/number states** $|n\rangle$

$$\boxed{\begin{aligned} \hat{H}|n\rangle &= \hbar\omega(n+\tfrac{1}{2})|n\rangle \\ \hat{N}|n\rangle &= n|n\rangle \end{aligned}}$$

Commutators

$$\left.\begin{aligned} [\hat{N}, \hat{a}^+] &= \hat{a}^+ \\ [\hat{N}, \hat{a}] &= -\hat{a} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}}$$

Generating excited states

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle}$$

Expectation values for $\hat{q}$ **and** $\hat{p}$ **in number states**

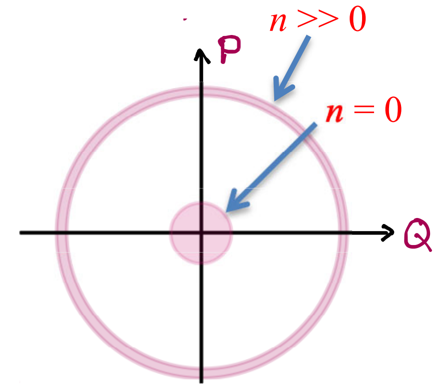
$$\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{q}^2|n\rangle = \frac{q_0^2}{2}(n+\tfrac{1}{2}) \neq 0$$

$$\langle n|\hat{p}^2|n\rangle = \frac{p_0^2}{2}(n+\tfrac{1}{2}) \neq 0$$

$$\Delta q\,\Delta p = \frac{q_0 p_0}{2}(n+\tfrac{1}{2}) = \hbar(n+\tfrac{1}{2})$$

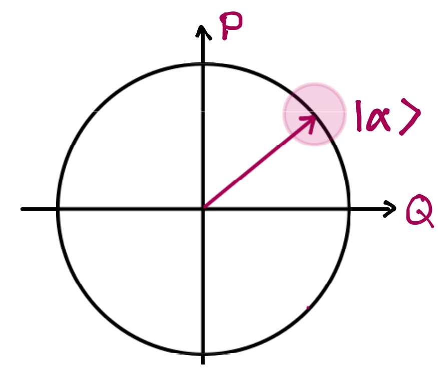
Phase space visualization of number states



Quasi-classical (coherent) state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_i \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\Delta q\,\Delta p = \hbar/2, \quad \Delta Q = \Delta P$$



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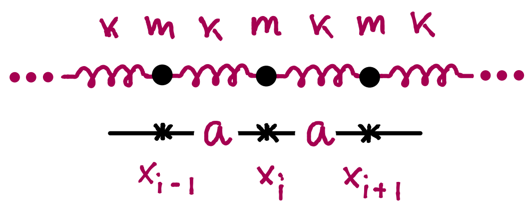
Lagrange formulation of 1D Scalar Field

"vibronium"



Configuration space = $\{x_i\}$ **(set of *N* osc. positions)**

$$T = \sum_{i=1} \tfrac{1}{2} m \dot{x}_i^2, \quad V = \sum_{i=1}^{N} \tfrac{1}{2} \kappa (x_{i+1} - x_i)^2$$

Lagrangian, equations of motion

Continuum limit ➡ Elastic rod

$N \to \infty$ $\quad m/a \to \mu$ ← **linear mass density**

$a \to dx$ $\quad \kappa a \to Y$ ← **Youngs modulus**

$\{x_i\} \to y(x)$ ← **displacement field (sound)**

Rewrite

$$\boxed{\begin{aligned} T &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2} \left(\tfrac{m}{a}\right) \dot{x}_i^2 = \int dx \tfrac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 \\ V &= \lim_{N\to\infty} \sum_{i=1}^{N} a \tfrac{1}{2} \kappa a \left(\frac{x_{i+1} - x_i}{a}\right)^2 = \int dx \tfrac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2 \end{aligned}}$$

Lagrangian:

$$\boxed{\mathcal{L} = T - V = \int dx \tfrac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 - \int dx \tfrac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2}$$

Notes, Homework ➡ Scalar wave equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} - \frac{Y}{\mu} \frac{\partial^2 y}{\partial x^2} = 0}$$

– Not yet ready for Quantization –

Introduction to Quantum Field Theory

Normal Mode Decomposition

Field in cavity:

Solutions to wave eq.

Let $y(x,t) = g(t)u(x) = g_0 e^{i\omega t} u(x)$

$$\ddot{y} - v^2 y'' = -\omega^2 g(t) u(x) - v^2 g(t) u''(x) = 0$$

$$\boxed{u''(x) = -k^2 u(x), \quad k = \omega/v}$$

Solutions in cavity:

$$\boxed{u_k(x) = \sqrt{\frac{2}{L}} \sin(kx), \quad k = \frac{n\pi}{L}}$$

These standing waves are a set of Normal Modes

These modes are orthonormal and complete

$$\boxed{y(x,t) = \sqrt{L} \sum_k q_k(t)\, u_k(x)}$$

Normal mode expansion of $y(x,t)$ **in basis** $u_k(x)$

Lagrangian for the acoustic field:

$$T = \int dx\, \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 = \sum_{k,k'} \frac{1}{2} \underbrace{\mu L}_{M}\, \dot{q}_k \dot{q}_{k'} \underbrace{\int dx\, u_k(x) u_{k'}(x)}_{\delta_{kk'}}$$

$$= \sum_k \frac{1}{2} M \dot{q}_k^2$$

$$V = \int dx\, \frac{1}{2} Y \left(\frac{\partial y}{\partial x}\right)^2 = \sum_{k,k'} \frac{1}{2} Y L\, q_k q_{k'} \int dx \left(\frac{\partial u_k}{\partial x}\right)\left(\frac{\partial u_{k'}}{\partial x}\right)$$

$$= \sum_k \frac{1}{2} M \omega_k^2 q_k^2$$

$$\boxed{\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 - \frac{1}{2} M \omega_k^2 q_k^2\right) = \sum_k \mathcal{L}_k}$$

Introduction to Quantum Field Theory

Jump-off Point for Todays Lecture

$$\mathcal{L} = T - V = \sum_k \left(\frac{1}{2} m \dot{q}_k^2 - \frac{1}{2} m \omega_k^2 q_k^2 \right) = \sum_k \mathcal{L}_k$$

Canonical Momentum

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = m \dot{q}_k$$

Hamiltonian

$$\mathcal{H}(\{p_k, q_k\}) = T + V = \sum_k \left(\frac{p_k^2}{2m} + \frac{1}{2} m \omega_k^2 q_k^2 \right)$$

(collection of SHO's, one for each normal mode)

Following the standard recipe…

$E_{0,k} = \hbar \omega_k$, $\quad q_{0,k} = \sqrt{2\hbar / m\omega_k}$, $\quad p_{0,k} = \sqrt{2m\hbar\omega_k}$

$Q_k = q_k / q_{0,k}$, $\quad P_k = p_k / p_{0,k}$, $\quad \alpha_k = Q_k + i P_k$

… we get solutions

$$\alpha_k(t) = Q_k(t) + i P_k(t) = \alpha_k(0)\, e^{-i\omega_k t}$$

This finally gives us

$$\mathcal{H} = \sum_k \hbar \omega_k (Q_k^2 + P_k^2) = \sum_k \hbar \omega_k \alpha_k^* \alpha_k$$

$$y(x,t) = \sqrt{L} \sum_k q_k(t) u_k(x)$$

$$= \frac{1}{2} \sum_k \sqrt{L q_{0,k}^2} \left(\alpha_k(t) u_k(x) + \alpha_k^*(t) u_k^*(x) \right)$$

allows real or complex $u_k(x)$

Introduction to Quantum Field Theory

Formal Quantization Procedure:

$$q_k \to \hat{q}_k, \quad p_k \to \hat{p}_k, \quad a_k \to \hat{a}_k$$

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar\,\delta_{kk'}, \quad [\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}] = 0$$

Note: $k \neq k'$ ⇨ operators commute

(normal modes = independent degs. of freedom)

Hamiltonian & Quantized fields

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}^\dagger_k \hat{a}_k + 1/2\right)$$

$$\hat{\eta}(x) = \sqrt{L}\sum_k \hat{q}_k\, u_k(x) = \sum_k \sqrt{L\, q^2_{0,k}}\left(\hat{a}_k u_k(x) + \hat{a}^\dagger_k u^*_k(x)\right)$$

$$\hat{\pi}(x) = \frac{1}{\sqrt{L}}\sum_k \hat{p}_k u_k(x) = -i\sum_k \sqrt{\frac{p^2_{0,k}}{L}}\left(\hat{a}_k u_k(x) - \hat{a}^\dagger_k u^*_k(x)\right)$$

field $\hat{\eta}(x)$ and canonical momentum field $\hat{\pi}(x)$

⇨ $[\hat{\eta}(x), \hat{\pi}(x')] = i\hbar\,\delta(x-x')$

Quantum States:

The quantum field is a collection of QSHO's

⇩

State space = tensor product of SHO spaces

Fock Space $\mathcal{E} = \mathcal{E}_{k_1} \otimes \mathcal{E}_{k_2} \otimes \mathcal{E}_{k_3} \otimes \ldots, \mathcal{E}_{k_j}$ ← **SHO space**

Fock State $|\{n_{k_1}, n_{k_2}, \ldots\}\rangle = |n_{k_1}\rangle \otimes |n_{k_2}\rangle \otimes \ldots |n_{k_j}\rangle$ ← **SHO state**

$\hat{a}_{k_i}, \hat{a}^\dagger_{k_i}$ **destroy/create excitations in mode k_i**

Vacuum State $|0\rangle$ ← **zero quanta in every mode**

Favorite Question: What is a Phonon?

Introduction to Quantum Field Theory

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0|\hat{y}(x)|0\rangle =$$

$$\sum_k \tfrac{1}{2}\sqrt{L q_{0,k}^2}\left(\langle 0|\hat{a}_k|0\rangle u_k(x) + \langle 0|\hat{a}_k^\dagger|0\rangle u_k^*(x)\right) = 0$$

Zero Point Fluctuations

$$\langle 0|\hat{y}(x)^2|0\rangle$$

$$= \sum_k \tfrac{1}{2} L q_{0,k} q_{0,k} \langle 0|\hat{a}_k^\dagger \hat{a}_k + 1|0\rangle u_k(x) u_k^*(x)$$

$$= \sum_k \tfrac{1}{2} L q_{0,k} |u_k(x)|^2 \neq 0$$

$$\Sigma_{kk'}\langle 0|(\hat{a}_k u_k + \hat{a}_k^\dagger u_k^*)(\hat{a}_{k'} u_{k'} + \hat{a}_{k'}^\dagger u_{k'}^*)|0\rangle =$$

$$\Sigma_k \langle 0|(\cancel{\hat{a}_k \hat{a}_k} u_k u_k + \hat{a}_k \hat{a}_k^\dagger u_k u_k^* + \cancel{\hat{a}_k^\dagger \hat{a}_k} u_k^* u_k + \cancel{\hat{a}_k^\dagger \hat{a}_k^\dagger} u_k^* u_k^*)|0\rangle =$$

$$\Sigma_k \langle 0|\hat{a}_k \hat{a}_k^\dagger u_k u_k^*|0\rangle = \Sigma_k \langle 0|(\hat{a}_k^\dagger \hat{a}_k + 1)|0\rangle u_k u_k^*$$

$$\hat{y}(x) = \sum_k \sqrt{L q_{0,k}^2}\left(\hat{a}_k u_k(x) + \hat{a}_k^\dagger u_k^*(x)\right)$$

Vacuum Fluctuations:

Expectation value of the Field

$$\langle 0|\hat{y}(x)|0\rangle =$$

$$\sum_k \tfrac{1}{2}\sqrt{L q_{0,k}^2}\left(\langle 0|\hat{a}_k|0\rangle u_k(x) + \langle 0|\hat{a}_k^\dagger|0\rangle u_k^*(x)\right) = 0$$

Zero Point Fluctuations

$$\langle 0|\hat{y}(x)^2|0\rangle$$

$$= \sum_k \tfrac{1}{2} L q_{0,k} q_{0,k} \langle 0|\hat{a}_k^\dagger \hat{a}_k + 1|0\rangle u_k(x) u_k^*(x)$$

$$= \sum_k \tfrac{1}{2} L q_{0,k} |u_k(x)|^2 \neq 0$$

Thus $\Delta y(x) \neq 0$ **with zero phonons in field**

Note the famous divergence:

$$E_{vac} = \langle 0|\hat{H}|0\rangle = \sum_k \frac{\hbar\omega_k}{2} \to \infty \text{ for } k \to \infty$$

Introduction to Quantum Field Theory

Are our Phonons waves or particles?

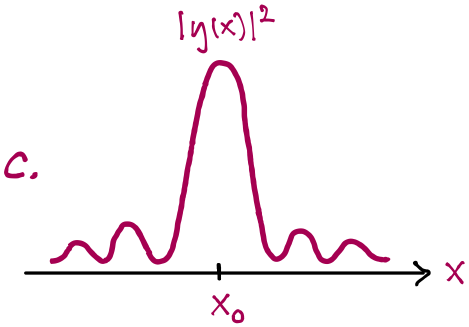
Extended

Localized

Particle-like Phonons

Classical Wavepacket

$$y(x) = \sum_k f_k u_k(x) + c.c.$$



Define $\hat{A}^+ = \sum_k f_k \hat{a}_k^+ , \quad \sum_k |f_k|^2 = 1$

$$\Rightarrow \hat{A}^+|0\rangle = f_{k_1}|1_{k_1}, 0_{k_2}, \ldots\rangle + f_{k_2}|0_{k_1}, 1_{k_2}, \ldots\rangle + \ldots$$

Localized excitation in the field.

These Particle-like Phonons are Bosons

$$\hat{A}'^+\hat{A}^+|0\rangle = \hat{A}^+\hat{A}'^+|0\rangle$$

1st particle @ x
2nd particle @ x'

1st particle @ x'
2nd particle @ x