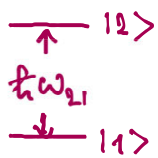
Atom-Light Interaction: 2-Level Approximation

General observation:

- Atoms and molecules often behave as if they have a single, dominant transition frequency
- We expect this when the freq. of the driving is resonant with one transition $|\varphi_n\rangle \to |\varphi_m\rangle$ and far off resonance with all others.

Interaction $\boxed{V_{ext} = -\hat{\vec{\mu}} \cdot \vec{E}(t)}$

State space $\mathcal{D}im(\mathcal{E}) = 2, \ \{|1\rangle, |2\rangle\}$



State vector

$$|\psi(t)\rangle = a_1(t)\,|1\rangle + a_2(t)\,|2\rangle$$

Schröd. eq.

$$i\hbar \dot{a}_1 = E_1 a_1 + V_{11} a_1 + V_{12} a_2$$
$$i\hbar \dot{a}_2 = E_2 a_2 + V_{21} a_1 + V_{22} a_2$$

$$V_{12}(t) = -\vec{\mu}_{12} \cdot \tfrac{1}{2}\left(\hat{\varepsilon} E_0 e^{-i\omega t} + c.c.\right)$$
$$V_{21}(t) = -\vec{\mu}_{21} \cdot \tfrac{1}{2}\left(\hat{\varepsilon} E_0 e^{-i\omega t} + c.c.\right)$$

Parity selection rule

Definition: $\vec{r} \to -\vec{r}$ is a reflection through the origin

Atomic Hamiltonian $H \propto \frac{1}{r}$ ⇨ $H(\vec{r}) = H(-\vec{r})$

⇨ Eigenstates $\varphi(\vec{r}) = \pm\, \varphi(-\vec{r}) = \varphi(-[-\vec{r}])$

"+" for even parity
"-" for odd parity

two reflections equals the identity

The dipole $\hat{\vec{\mu}}$ is a vector operator ⇨

transforms like a vector when $\vec{r} \to -\vec{r}$

Thus $\hat{\vec{\mu}}(\vec{r}) = e\hat{\vec{r}} = -\hat{\vec{\mu}}(-\hat{\vec{r}})$ and

$\vec{\mu}_{nm} = \int d^3r\, \varphi_n^*(\vec{r})\, \vec{\mu}\, \varphi_m(\vec{r}) \neq 0$ only when

φ_n and φ_m have ***opposite parity***

Parity rule: No dipole moment in energy eigenstate !

$$\boxed{\begin{array}{l} \vec{\mu}_{12} = \langle 1|\hat{\vec{\mu}}|2\rangle, \quad \vec{\mu}_{21} = \vec{\mu}_{12}^* \\ \vec{\mu}_{11} = \vec{\mu}_{22} = 0 \ \Rightarrow \ V_{11} = V_{22} = 0 \end{array}}$$

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We define

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}, \qquad E_1 = 0$$

$$\left.\begin{aligned} \chi_{12} &= \vec{\mu}_{12}\cdot\hat{\varepsilon}E_0/\hbar \\ \chi_{21} &= \vec{\mu}_{21}\cdot\hat{\varepsilon}E_0/\hbar \end{aligned}\right\}$$

interaction energy is $\hbar\chi$

χ : Rabi frequency

Note:

$$\begin{cases} \chi_{12}^{*} = \vec{\mu}_{21}\cdot(\hat{\varepsilon}E_0/\hbar)^{*} \neq \chi_{21} \\ \chi_{21}^{*} = \vec{\mu}_{12}\cdot(\hat{\varepsilon}E_0/\hbar)^{*} \neq \chi_{12} \end{cases}$$

Plug into $i\hbar\dot{\underline{a}} = \underline{\underline{H}}_a\,\underline{a} + \underline{\underline{V}}\,\underline{a}$ (S. E.) to get

$$i\dot{a}_1 = -\frac{1}{2}\left(\chi_{12}e^{-i\omega t} + \chi_{21}^{*}e^{i\omega t}\right)a_2$$
$$i\dot{a}_2 = \omega_{21}a_2 - \frac{1}{2}\left(\chi_{21}e^{-i\omega t} + \chi_{12}^{*}e^{i\omega t}\right)a_1$$

Switch to rotating frame (slow variables)

$$c_1(t) = a_1(t), \qquad c_2(t) = a_2(t)\,e^{i\omega t}$$

$$i\dot{c}_1(t) = -\frac{1}{2}\left(\chi_{12}e^{-i2\omega t} + \chi_{21}^{*}\right)c_2(t)$$
$$i\dot{c}_2(t) = (\omega_{21}-\omega)c_2(t) - \frac{1}{2}\left(\chi_{21} + \chi_{12}^{*}e^{i2\omega t}\right)c_2(t)$$

Rotating Wave Approximation (RWA)

Very important, equivalent to resonant approximation

Terms $\propto e^{\pm i2\omega t}$ average to zero on time scale for variations in c_1, c_2

$$i\dot{c}_1(t) = -\frac{1}{2}\chi_{21}^{*}c_2(t)$$
$$i\dot{c}_2(t) = \Delta c_2(t) - \frac{1}{2}\chi_{21}c_1(t)$$

$\Delta = \omega_{21} - \omega$

(detuning)

Exactly Solvable !

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To simplify further, make a global phase choice such that $\chi_{21} = \vec{\mu}_{21} \cdot \hat{\varepsilon} E_0 / \hbar = \chi$ is real (not required)

Simplest 2-level equations

$$i\dot{C}_1(t) = -\frac{1}{2}\chi C_2(t)$$

$$i\dot{C}_2(t) = \Delta C_2(t) - \frac{1}{2}\chi C_1(t)$$

Homework: Show that for $C_1(0) = 1,\ C_2(0) = 0$

$$C_1(t) = \left(\cos\frac{\Omega t}{2} + i\frac{\Delta}{\Omega}\sin\frac{\Omega t}{2}\right) e^{-i\Delta t/2}$$

$$C_2(t) = \left(i\frac{\chi}{\Omega}\sin\frac{\Omega t}{2}\right) e^{-i\Delta t/2}$$

These are the **Rabi Solutions**

χ: Rabi freq.

Δ: Detuning

$\Omega \equiv \sqrt{\chi^2 + \Delta^2}$: Generalized Rabi freq.

Note: The Rabi Solutions give us the entire state, in the lab (a's) and rotating (c's) frames

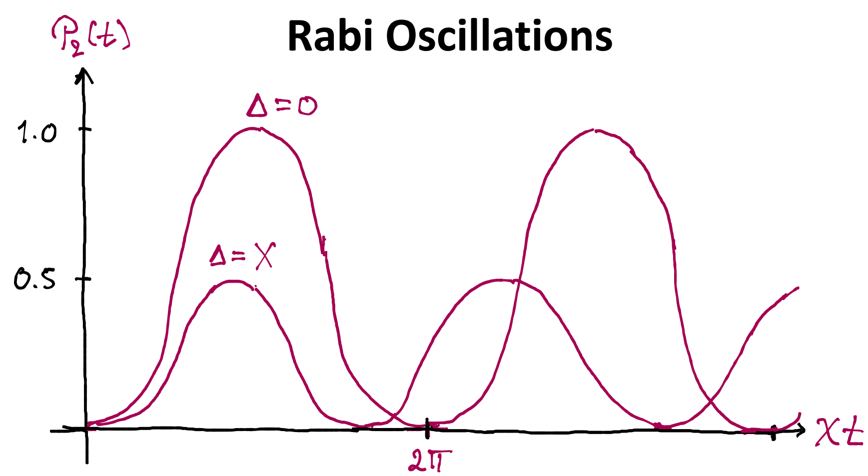
We have maximum information about the system and can make any predictions allowed by QM

Probabilities of finding the atom in states $|1\rangle, |2\rangle$

$$P_1(t) = |C_1(t)|^2 = \frac{1}{2}\left(1 + \frac{\Delta^2}{\Omega^2}\right) + \frac{1}{2}\frac{\chi^2}{\Omega^2}\cos\Omega t$$

$$P_2(t) = |C_2(t)|^2 = \frac{1}{2}\frac{\chi^2}{\Omega^2}\left(1 - \cos\Omega t\right)$$

Rabi Oscillations



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Note: All 2-level systems are **isomorphic**

- Equivalent Observables
- Equivalent Phenomena
- The Rabi problem was first solved in **ESR** and **NMR**, for spin-1/2 particles with a magnetic moment $\vec{\mu}$ driven by a magnetic field $\vec{B}$ with interaction $H = \vec{\mu} \cdot \vec{B}$
- 2-level systems are now often called **qubits**

Dressed States

The 2-level eqs. in the RWA look like a S.E. with

$$\boxed{H_{RWA} = \hbar \begin{pmatrix} 0 & \frac{1}{2}\chi \\ -\frac{1}{2}\chi & \Delta \end{pmatrix}}$$

The eigenstates of H_{RWA} are called **Dressed States**

The DS are stationary only in the Rotating Frame.

In the Lab Frame (Schrödinger Picture) they are periodic, oscillating w/frequency ω

Comparison to the Classical Lorentz atom

Goal: To understand why the Lorentz model works so well, and to determine its limits of validity

Classical Equation of Motion:

$$\boxed{\left(\frac{d^2}{dt^2} + \omega_0^2\right)\vec{r} = \frac{e}{m}\vec{E}}$$

will derive similar equation for $\langle\hat{\vec{r}}\rangle$

Equation of Motion for $\langle\hat{\vec{r}}\rangle$. First step:

$$\langle\hat{\vec{r}}\rangle = \langle\psi|\hat{\vec{r}}|\psi\rangle = (a_1^*, a_2^*)\begin{pmatrix} 0 & \vec{r}_{12} \\ \vec{r}_{21} & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= a_1^* a_2 \vec{r}_{12} + a_2^* a_1 \vec{r}_{21} = \vec{r}_{12}(a_1^* a_2 + a_2^* a_1)$$

(choose phase to make $\vec{r}_{12}$ real)

We need an expression for $\frac{d^2}{dt^2}\langle\hat{\vec{r}}\rangle$

We can find it from the S. E., i. e., the eqs for the *a*'s back when we first set up the Rabi problem.

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We pick linear polarization so $\vec{\mathcal{E}} E_0$ is real-valued and $V_{12} = V_{21} = V$. In that case the eqs for the *a*'s are

$$\hbar \dot{a}_1^* = i\left(E_1 a_1^* + V a_2^*\right)$$

$$\hbar \dot{a}_2 = -i\left(E_2 a_2 + V a_1\right)$$

With this we have

$$\frac{d}{dt} a_1^* a_2 = \left(\dot{a}_1^* a_2 + a_1^* \dot{a}_2\right)$$

$$= -i \underbrace{\frac{E_2 - E_1}{\hbar}}_{\omega_0} a_1^* a_2 - i \frac{V}{\hbar}\left(|a_1|^2 - |a_2|^2\right)$$

Differentiating again gives us

$$\frac{d^2}{dt^2}\left(a_1^* a_2\right) = -\omega_0^2 a_1^* a_2 - \frac{\omega_0 V}{\hbar}\left(|a_1|^2 - |a_2|^2\right)$$

$$- i\hbar \frac{d}{dt}\left[\frac{V}{\hbar}\left(|a_1|^2 - |a_2|^2\right)\right]$$

Looking at the eq. for $\langle \hat{\vec{\wp}} \rangle$ suggests we should add the complex conjugate and multiply w/ $\vec{\wp}_{12}$

This gives us

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right)\langle \hat{\vec{\wp}} \rangle = \frac{2\omega_0 \vec{\wp}_{12} V}{\hbar}\left(|a_1|^2 - |a_2|^2\right)$$

$$= \frac{2\omega_0}{\hbar} \vec{\wp}_{12}\left(\vec{\wp}_{12} \cdot \vec{E}\right)\left(|a_1|^2 - |a_2|^2\right)$$

$\vec{\wp}_{12} = \langle 1 | \hat{\vec{\wp}} | 2 \rangle$: dipole matrix element

To wrap up, we need to know a bit about real, multilevel atoms. (We will revisit this soon)

Pick linear polarization so $\vec{\mathcal{E}}$ is real-valued.
Pick quantization axis along $\vec{\mathcal{E}}$ $\Rightarrow$ $\vec{\wp}_{12} = \wp_{12} \vec{\mathcal{E}}$

$$\boxed{\left(\frac{d^2}{dt^2} + \omega_0^2\right)\langle \hat{\vec{\wp}} \rangle = \frac{2\omega_0 \wp_{12}^2}{\hbar} \vec{E}\left(|a_1|^2 - |a_2|^2\right)}$$

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Compare to Classical Equation of Motion

$$\boxed{\left(\frac{d^2}{dt^2} + \omega_0^2\right)\vec{p} = \frac{e}{m}\vec{E}}$$

The two eqs. have the same form if $|a_1|^2 \sim 1$, $|a_2|^2 \sim 0$

This is the case for $\Delta \gg \chi$

Or when $\chi \ll \Gamma$

Γ: Decay rate of $|2\rangle$

Limit of weak Excitation !

Oscillator Strength $\boxed{f = \frac{2m\omega_0}{\hbar e^2}\, p_{12}^2}$

$$\boxed{\left(\frac{d^2}{dt^2} + \omega_0^2\right)\langle\hat{p}\rangle = \frac{e^2}{m} f \vec{E}}$$

Exactly like the classical equation, but with modified polarizability !