1. Absorption and thermal distributions in a 2-level system

Consider a collection of identical two-level atoms in thermal equilibrium. The population distribution between the two states $\psi_g$ (the ground state) and $\psi_e$ (the excited state) is given by the Maxwell-Boltzmann distribution (as discussed in class, and given by equation 3.6.7 in the "Laser Physics" textbook).

(a) With $N_g$ and $N_e$ denoting population densities corresponding to states $\psi_g$ and $\psi_e$, calculate $N_e/N_g$ for an optical transition of wavelength $\lambda = 500 \text{ nm}$ and a temperature of 300 K. ($\hbar = 1.1 \times 10^{-34} \text{ J s}$, and $k_B = 1.4 \times 10^{-23} \text{ J/K}$).

(b) At what temperature will $N_e$ be equal to 0.01$N_g$?

(c) Is there any realistic temperature for which $N_e$ can exceed $N_g$ for this two-level system?

(d) Can a gas of two-level atoms held at any temperature provide more gain than absorption to a beam of light that passes through the gas? Why or why not?

2. Absorption lineshapes

(a) The HeNe gain profile for the $\lambda = 633 \text{ nm}$ transition is known to be Doppler broadened to $\delta \nu_D \approx 1500 \text{ MHz}$ for $T=400\text{K}$. Calculate the Doppler broadened absorption line width for the 1S-2P transition in hydrogen assuming you have a gas cell of hydrogen atoms at room temperature.

(b) Calculate the natural absorption line width for the 1S-2P transition in hydrogen and determine what the line shape for this transition would look like if measured in the gas cell from part (a) (i.e. Gaussian or Lorentzian?).
Absorption in a 2-level system

The absorption coefficient $a$ is related to many other concepts in optical physics, such as laser cooling, the optical dipole force, and of course optical gain via stimulated emission when a population inversion exists. In this assignment, many terms and definitions are reviewed and some introduced.

In class, we have seen a general expression for the absorption coefficient may be written as:

$$a = \Delta N \sigma(\omega),$$

where the population density difference between levels 1 and 2 is given by $\Delta N = N_1 - N_2$ and $\omega = 2\pi\nu$. Since $a$ has units of inverse length, the new variable $\sigma$ must have units of area per atom. Specifically, $\sigma$ is interpreted as the cross-sectional area for an atom-photon interaction. If we think loosely (and incorrectly) of a photon as a particle propagating through space, then the bigger $\sigma$ is, the “bigger” the photon and atom look to each other. For a given photon flux and atomic density, a larger $\sigma$ will mean that an absorption event is more likely to occur.

For a homogeneously broadened medium, we have seen that a general expression for the absorption (or stimulated emission) cross-section can be written as:

$$\sigma(\nu) = \frac{\lambda^2}{8\pi} A_{21} S(\nu) = \sigma_0 \times \frac{\delta\nu_{rad}}{\delta\nu_H} \times \frac{1}{1 + (\Delta/\delta\nu_H)^2},$$

where $S(\nu)$ is the line shape function, $\Delta = (\nu - \nu_0)$ is the usual light detuning from the atomic resonance, $\sigma_0 = \lambda_0^2/(2\pi)$ is the maximum absorption cross-section, and $\lambda_0$ is the wavelength of the atomic resonance ($\lambda_0 = 2\pi c/\omega_0$). The full-width at half maximum (FWHM) of this homogeneous Lorentzian lineshape is given by $2 \times \delta\nu_H = 2 \times (\delta\nu_0 + \delta\nu_{rad}) = A_{21}/(2\pi) + \gamma_c/\pi$, where $\gamma_c$ is the elastic collision rate of the atoms. Note that if this were a multilevel atom and level 1 was not the ground state, the natural line width (which is due to spontaneous emission alone) would more generally be given by $2 \times \delta\nu_{rad} = (A_1 + A_2)/2\pi$, where $A_2 = \sum A_{2i}$ and $A_1 = \sum A_{1j}$ are the total radiative decay rates from each level to all other allowed states. The Einstein $A$ coefficient is given by the usual expression: $A_{21} = \omega_0^3 \nu^2/(3\pi\epsilon_0 \hbar c^3)$.

**3. Saturation.** In this problem and the following problems, we will consider the steady-state absorption coefficient for a strictly two-level atom (e.g. no spontaneous emission from the lower level) interacting with a monochromatic polarized beam of laser light, and we assume that the gas is not Doppler-broadened (i.e. homogeneous absorption), that state $\psi_1$ is the ground state, and state $\psi_2$ is the excited state. We will also assume that collisional broadening is negligible, and therefore $\delta\nu_H = A_{21}/4\pi = \delta\nu_{rad}$ for this strictly 2-level system.

(a) Set up the population rate equations for this 2-level system and solve for the steady-state population density for level 2, $N_{2ss}$, in terms of only the spontaneous emission rate $A_{21}$, the stimulated transition rate $R(I)$, and the total population density $N = N_1 + N_2$.

(b) Using the same procedure, solve for $N_{1ss}$ and obtain an expression for the steady-state population difference $\Delta N$. Recalling that the stimulated transition rate can be written as $R(I) = I \times (\sigma/\hbar\omega_0)$, show that $\Delta N$ can be expressed in the form:

$$\Delta N = \frac{\Delta N^o}{1 + I/I_{sat}},$$

where $I_{sat} = \hbar\omega_0/(2\sigma\tau_2)$, $\tau_2$ is the natural lifetime of level 2, and $\Delta N^o$ is the initial population difference with the incident light off ($I = 0$). Recall that for a 2-level system in thermal equilibrium,
the resulting thermal distribution is $N_2^0/N_1^0 = e^{-\hbar\omega_0/k_BT}$, so that $N_1^0 >> N_2^0$. Thus in this particular system $N_1^0 - N_2^0 \approx N_1^0$, which is approximately the total population density $N$ of the gas, with units of number of atoms per unit volume.

(c) We can now write the absorption coefficient as:

$$a(\nu) = \frac{a_o(\nu)}{1 + \frac{I}{I_{sat}}}$$

where we have defined the small-signal absorption coefficient for this 2-level system as $a_o(\nu) = \sigma(\nu)N$.

Show that the absorption coefficient can now be put into the form:

$$a(\nu) = N \times \sigma_o \times \frac{1}{1 + \frac{I}{I_{sat}^o} + (\Delta/\delta\nu_H)^2}$$

where $I_{sat}^o = \hbar\omega_0/(2\sigma_o\tau_2)$. Note that $I_{sat}^o$ is a constant while $I_{sat}$ (defined above) depends on detuning from resonance due to $\sigma(\nu)$. We can therefore write:

$$I_{sat} = I_{sat}^o \times [1 + (\Delta/\delta\nu_H)^2]$$

The meaning of $I_{sat}$ is discussed in class and is hopefully now clear; it is the intensity at which the initial population difference is reduced to half its value. If $\Delta$ is not zero, a higher intensity is needed to saturate the transition. Note also that the expression for $I_{sat}$ can vary by an overall factor depending on the whether one is considering a 2-level system or a different multilevel system (for example, if the lower level is not the ground state for the atom).

(d) Make a plot of $a/(N\sigma_o)$ vs. $I/I_{sat}^o$ for case $\Delta = 0$. We had previously seen that for a gas of two-level atoms in thermal equilibrium, a population inversion ($N_2 > N_1$) can not be achieved. By now, you should be convinced that under steady-state conditions, even a high-intensity laser can not invert a sample of two-level atoms. Thus a gas of two-level atoms is always an absorber.

(e) Finally, show that $a$ can be written in the following form:

$$a = \left(\frac{\delta\nu_H}{\delta\nu'}\right)^2 \times \frac{a_o(\nu_o)}{1 + (\Delta/\delta\nu')^2},$$

where $\delta\nu' = \delta\nu_H\sqrt{1 + I/I_{sat}^o}$ is the power broadened line width. On the same graph, make a plot of $a/(N\sigma_o)$ versus detuning $\Delta$ for $I = 0, I = I_{sat}^o$, and $I = 3I_{sat}^o$. 

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4. Photon scattering. In this problem we will consider the same two-level system as in problem 3. In thermal equilibrium, with only blackbody radiation present, at any instant in time the vast majority of atoms in a gas will reside in the electronic ground state $\psi_1$. In steady-state equilibrium after near-resonant light has been turned on, each atom will then have a higher probability of being found in the excited state, due to interaction with the light.

The rate of photon scattering by a gas of atoms (i.e. the absorption and spontaneous emission of a photon from the beam of light) is an important quantity, and is useful to know in many applications, such as determining the amount of fluorescence observed as a laser beam passes through a gas of atoms. This scattering rate per atom ($\gamma_{scat}$) depends on the steady-state value for atoms in level 2 and the spontaneous emission rate from level 2:

$$\gamma_{scat} = A_2 \times \frac{N^{ss}_2}{N}.$$  

(a) Using your result from problem 3(a) for $N^{ss}_2$, show that $\gamma_{scat}$ can then be written as

$$\gamma_{scat} = \frac{A_2}{2} \cdot \left( \frac{s_0}{1 + s_0} \right) \left( \frac{1}{1 + \Delta^2/\delta\nu_H^2} \right),$$

where $s_0 = I/I_{sat}$ is the on-resonance saturation parameter, and $\delta\nu' = \delta\nu_H \sqrt{1 + s_0}$.

In an experiment, you might imagine sending a tunable laser beam through a gas of atoms, collecting some of the fluorescence with a lens, and measuring the amount of this scattered light on a photodiode. As you scan the laser frequency through the atomic resonance, you’ll see a power-broadened Lorentzian lineshape (as long as Doppler broadening can be neglected) with a full width at half maximum (FWHM) of $2 \times \delta\nu_{rad} = A_2/(2\pi)$. The peak scattering rate (when $\Delta = 0$) is given by $\gamma_{scat} = (A_2/2)^s_0$, so the amount of scattered light will increase with $I$, but will saturate to a limiting value as $I$ far exceeds $I_{sat}$.

(b) Calculate $\sigma_0$ for an atomic transition with a resonance at a wavelength of 780 nm.

(c) Calculate $I_{sat}^o$ for an atomic transition with a FWHM linewidth of $A_2/(2\pi) = 6$ MHz.

(d) Suppose you let a confined gas of atoms interact with a resonant ($\Delta = 0$) laser beam with uniform intensity $I = I_{sat}$. Assume that $10^9$ atoms are contained in a very tiny volume, approximately a single point in space. As the atoms interact with the light, they uniformly scatter photons into all directions at a rate $\gamma_{scat}$. A fraction $\delta$ of this light is collected by a lens of both diameter and focal length 5 cm, placed 5 cm away from the fluorescing atoms, and the collected light is focused onto a photodiode. If the atoms have a FWHM linewidth of 6 MHz, what is the power (in Watts) that is incident on the photodiode? [Detected Power = (fraction of total scattered light collected) x (energy per photon) x (number photons scattered per atom per unit time) x (number of atoms in the sample).]
Population inversion and gain in a 3-level system

When discussing gain rather than absorption, the population difference is defined as

\[ \Delta N = N_2 - N_1, \]

and the gain coefficient is

\[ g(\nu) = \Delta N \sigma(\nu). \]

The saturation of the gain is again due to the change in \( \Delta N \) that results from stimulated transitions between levels 1 and 2. It is hopefully now clear from the previous problems that one cannot create a population inversion in a strictly 2-level system. The expression for \( \Delta N \) depends on the particular multilevel system being studied. Most laser systems can be modeled by reducing them to an effective 3 or 4 level laser system. In this problem, we will assume there is no collisional or Doppler broadening.

5. (a) Write the rate equations for a 3-level atomic system as discussed in class, assuming the usual conditions and notation: (i) the lasing transition is between levels 1 and 2, where level 1 is the ground level, (ii) pumping is from level 1 to 3 at a rate per atom \( P \), and (iii) instantaneous decay occurs from level 3 to 2.

(b) Solve the rate equations for steady state population densities \( N_1 \) and \( N_2 \). However, do not assume that field intensities are small. Keep the absorption and stimulated emission rate \( R(I) = I \cdot \sigma/(\hbar \omega_0) \) in your equations. Be sure \( N_1 \) and \( N_2 \) are expressed only in terms of \( P, R(I), A_{21}, \) and \( N \), the total atom number density.

(c) Using your result above, solve for \( \Delta N \).

(d) Setting \( R(I) \sim 0 \), solve for \( \Delta N^0 \) and show that it matches the expression:

\[ \Delta N^0 = N \times \frac{P - A_{21}}{P + A_{21}}. \]

(d) Finally, show that the population difference for a three level laser system can be written:

\[ \Delta N = \frac{\Delta N^0}{1 + I/I_{sat}}, \]

where the saturation intensity is now given by:

\[ I_{sat} = \frac{\hbar \omega_0}{2\sigma} \times (A_{21} + P). \]

Note that if \( P=0 \), the gain coefficient is negative (absorption) and the saturation intensity is the same as previously expressed for the two level system.
6. **Scattering force.** Consider a beam of light of frequency $\omega$ and wavevector $\vec{k}$. The beam is incident on a gas of two-level atoms. As a ground-state atom absorbs a single photon from the beam, the atom gets a kick of momentum $\vec{p}_p = \hbar \vec{k} = \hbar |k| \hat{k}$. When the atom returns to its ground state by spontaneous emission, it also receives a momentum kick, this time in the direction opposite to the direction of the emitted photon, which is assumed to be in any random direction as is usually appropriate for spontaneous emission. If an atom scatters photons at a rate $\gamma_{\text{scat}}$, all of the absorption events produce a net force in the $\hat{k}$ direction. The emission events, which occur in random directions, average to a net force of zero. Thus, the total **scattering force** of light on atoms is given by

$$\vec{F} = \frac{d\vec{p}}{dt} = \hbar \vec{k} \gamma_{\text{scat}} = \hat{k} \cdot \hbar |k| A_{21}^2 \frac{I/I_{\text{sat}}}{1 + I/I_{\text{sat}}^o + 4(\Delta/\Gamma)^2}.$$ 

**Doppler cooling** utilizes the scattering force to remove kinetic energy from a gas of atoms via momentum removal from the atoms, thus cooling atoms. Consider an atom moving in the $-\hat{k}$ direction, opposite to the propagation direction of a laser beam. The scattering force will push the atom in the $+\hat{k}$ direction, and will thus slow the atom down. However, the Doppler effect must be considered when atomic motion is relevant. To effectively slow down and cool an atom, we should have a beam of light tuned below the atomic resonance: $\omega < \omega_0$. This is called **red detuning** (the light is closer to the low-frequency, or red, end of the spectrum). If the atom moves toward the beam with a velocity $-|v_k|$, the atom will see the beam’s frequency as being Doppler shifted closer to resonance by an amount approximately given by $|k_0 v_k|$. Thus, the atom will scatter light most effectively from this beam when it moves into the beam, and when the amount of detuning is close to the Doppler shift: $\Delta \sim |k_0 v_k|$.

If a second red-detuned beam, with wavevector $-\vec{k}$ is also incident on this atom, the atom will see this new light as being even further away from resonance (since the atom is moving in the same direction as this second beam), and the atom will scatter light from this beam much less effectively than from the first beam of light propagating in the $+\hat{k}$ direction. Thus, no matter what component of velocity the atom has along the $\hat{k}$ direction in this two-beam field, the atom will slow down due to unbalanced scattering forces that always oppose the direction of atomic motion. This concept is called Doppler cooling (one type of laser cooling). When 3 pairs of red-detuned laser beams are configured such that there is always a scattering force in any direction (3 orthogonal pairs of beams), atoms with a velocity vector in any direction can be cooled. There is however a lower limit on the temperatures that can be achieved by Doppler cooling: $T_D = \hbar A_{21}^2/(2k_B)$, where $k_B$ is Boltzmann’s constant. (This limit originates in the fact that when the atom has reached low enough temperatures and thus very small velocities and Doppler shifts, the scattering forces are roughly balanced, and the atom undergoes a random walk in momentum space and scatters light from all beams at nearly the same rate.)

(a) What is the Doppler-limit temperature $T_D$ for an atom with a linewidth of $A_{21} \approx 2\pi \times 6$ MHz? This number is appropriate for the primary excited states of lithium, sodium, potassium, rubidium, and cesium.

(b) What is the natural lifetime of these atomic excited states?

(c) What is the one-dimensional velocity $v_D$ of a rubidium-87 atom if it is in a gas of temperature $T = T_D$? Use $(1/2)m_{Rb}v_D^2 = (1/2)k_B T_D$, with $m_{Rb} = 1.5 \times 10^{-25}$ kg.

(d) What is the velocity of a rubidium atom at room temperature ($T = 300$ K)?

(e) What is the deBroglie wavelength of a rubidium atom for each of the two velocities found in (c) and (d)?
(f) Calculate \(2k_0\bar{u}/A_{21}\), the ratio of the Doppler-broadened linewidth to the natural linewidth, for a sample of rubidium atoms at temperature \(T = 300\) K, and for a sample of atoms at \(T = T_D\). In which of these cases is the transition Doppler broadened? Use the mean speed \(\bar{u} = \sqrt{2k_BT/m_{Rb}}\). Also, use \(k_0 = 2\pi/\lambda_0\), with \(\lambda_0 = 780\) nm, the main resonance wavelength for \(^{87}\text{Rb}\).

(g) Suppose an isolated sample of atoms is very cold, say \(T << T_D\). If a single beam of light at frequency \(\omega\) interacts with the gas, it will heat each atom at a rate of \(\sim \hbar\omega_0\gamma_{\text{scat}}\). If a single atom scatters \(X\) photons, it gains on average an amount \(X\cdot[p^2/(2m)]\) of kinetic energy, where \(p_p = \hbar k\) is the momentum of a photon. Thus the kinetic energy gain per photon scattering event is the **photon recoil energy**, \(E_{\text{rec}} = (\hbar k)^2/(2m)\), where \(k\) is the wave number of the applied light (not necessarily at atomic resonance). Calculate the photon recoil energy for a rubidium atom scattering resonant light. Also, express the answer in temperature units: \(T_{\text{rec}} = 2E_{\text{rec}}/k_B\) (which comes from \((1/2)k_BT_{\text{rec}} = E_{\text{rec}}\)). Each scattered photon thus raises an atom’s temperature an average of approximately \(T_{\text{rec}}\).

(h) About how many photons will each atom of a sample need to scatter in order to take a gas of atoms at \(T \sim T_{\text{rec}}\) up to the Doppler temperature \(T_D\) (i.e., what is the ratio \(T_D/T_{\text{rec}}\))?