OPTI 511, Spring 2016 **Problem Set 9** Prof. R. J. Jones

Due Friday, April 29

1. Absorption and thermal distributions in a 2-level system

Consider a collection of identical two-level atoms in thermal equilibrium. The population distribution between the two states ψ_g (the ground state) and ψ_e (the excited state) is given by the Maxwell-Boltzmann distribution (as discussed in class, and given by equation 3.6.7 in the "Laser Physics" textbook).

(a) With N_g and N_e denoting population densities corresponding to states ψ_g and ψ_e, calculate N_e/N_g for an optical transition of wavelength $\lambda = 500$ nm and a temperature of 300 K. ($\hbar = 1.1 \times 10^{-34}$ J s, and $k_B = 1.4 \times 10^{-23}$ J/K).

(b) At what temperature will N_e be equal to $0.01N_g$?

(c) Is there **any** realistic temperature for which N_e can *exceed* N_g for this two-level system?

(d) Can a gas of two-level atoms held at any temperature provide more gain than absorption to a beam of light that passes through the gas? Why or why not?

2. Absorption lineshapes

(a) The HeNe gain profile for the $\lambda = 633$ nm transition is known to be Doppler broadened to $\delta\nu_D \approx 1500$ MHz for T=400K. Calculate the Doppler broadened absorption line width for the 1S-2P transition in hydrogen assuming you have a gas cell of hydrogen atoms at room temperature.

(b) Calculate the *natural* absorption line width for the 1S-2P transition in hydrogen and determine what the line shape for this transition would look like if measured in the gas cell from part (a) (i.e. Gaussian or Lorentzian?).

Absorption in a 2-level system

The absorption coefficient a is related to many other concepts in optical physics, such as laser cooling, the optical dipole force, and of course optical gain via stimulated emission when a population inversion exists. In this assignment, many terms and definitions are reviewed and some introduced.

In class, we have seen a general expression for the absorption coefficient may be written as:

$$a = \Delta N \sigma(\omega),$$

where the population density difference between levels 1 and 2 is given by $\Delta N = N_1 - N_2$ and $\omega = 2\pi\nu$. Since α has units of inverse length, the new variable σ must have units of area per atom. Specifically, σ is interpreted as the cross-sectional area for an atom-photon interaction. If we think loosely (and incorrectly) of a photon as a particle propagating through space, then the bigger σ is, the "bigger" the photon and atom look to each other. For a given photon flux and atomic density, a larger σ will mean that an absorption event is more likely to occur.

For a homogeneously broadened medium, we have seen that a general expression for the absorption (or stimulated emission) cross-section can be written as:

$$\sigma(\nu) = \frac{\lambda^2}{8\pi} A_{21} S(\nu) = \sigma_o \times \frac{\delta\nu_{rad}}{\delta\nu_H} \times \frac{1}{1 + (\Delta/\delta\nu_H)^2},$$

where $S(\nu)$ is the line shape function, $\Delta = (\nu - \nu_o)$ is the usual light detuning from the atomic resonance, $\sigma_0 = \lambda_0^2/(2\pi)$ is the maximum absorption cross-section, and λ_0 is the wavelength of the atomic resonance ($\lambda_0 = 2\pi c/\omega_0$). The full-width at half maximum (FWHM) of this homogeneous Lorentzian lineshape is given by $2 \times \delta\nu_H = 2 \times (\delta\nu_o + \delta\nu_{rad}) = A_{21}/(2\pi) + \gamma_c/\pi$, where γ_c is the elastic collision rate of the atoms. Note that if this were a multilevel atom and level 1 was not the ground state, the natural line width (which is due to spontaneous emission alone) would more generally be given by $2 \times \delta\nu_{rad} = (A_1 + A_2)/2\pi$, where $A_2 = \sum A_{2i}$ and $A_1 = \sum A_{1j}$ are the *total* radiative decay rates from each level to all other allowed states. The Einstein A coefficient is given by the usual expression: $A_{21} = \omega_0^3 \wp^2/(3\pi\epsilon_0\hbar c^3)$.

3. Saturation. In this problem and the following problems, we will consider the steady-state absorption coefficient for a strictly two-level atom (e.g. no spontaneous emission from the lower level) interacting with a monochromatic polarized beam of laser light, and we assume that the gas is *not* Doppler-broadened (i.e. homogeneous absorption), that state ψ_1 is the ground state, and state ψ_2 is the excited state. We will also assume that collisional broadening is negligible, and therefore $\delta\nu_H = A_{21}/4\pi = \delta\nu_{rad}$ for this strictly 2-level system.

(a) Set up the population rate equations for this 2-level system and solve for the steady-state population density for level 2, N_2^{ss}, in terms of only the spontaneous emission rate A_{21}, the stimulated transition rate $R(I)$, and the total population density $N = N_1 + N_2$.

(b) Using the same procedure, solve for N_1^{ss} and obtain an expression for the steady-state population difference ΔN. Recalling that the stimulated transition rate can be written as $R(I) = I \times (\sigma/\hbar\omega_o)$, show that ΔN can be expressed in the form:

$$\Delta N = \frac{\Delta N^o}{1 + I/I_{sat}},$$

where $I_{sat} = \hbar\omega_0/(2\sigma\tau_2)$, τ_2 is the natural lifetime of level 2, and ΔN^o is the *initial* population difference with the incident light off ($I = 0$). Recall that for a 2-level system in thermal equilibrium, the resulting thermal distribution is $N_2^0/N_1^0 = e^{-\hbar\omega_0/k_BT}$, so that $N_1^0 >> N_2^0$. Thus in this particular system $N_1^0 - N_2^0 \approx N_1^0$, which is approximately the total population density N of the gas, with units of number of atoms per unit volume.

(c) We can now write the absorption coefficient as:

$$a(\nu) = \frac{a_o(\nu)}{1 + \frac{I}{I_{sat}}}$$

where we have defined the *small-signal absorption coefficient* for this 2-level system as $a_o(\nu) = \sigma(\nu)N$.

Show that the absorption coefficient can now be put into the form:

$$a(\nu) = N \times \sigma_o \times \frac{1}{1 + I/I_{sat}^o + (\Delta/\delta\nu_H)^2}$$

where $I_{sat}^o = \hbar\omega_0/(2\sigma_o\tau_2)$. Note that I_{sat}^o is a constant while I_{sat} (defined above) depends on detuning from resonance due to $\sigma(\nu)$. We can therefore write:

$$I_{sat} = I_{sat}^o \times [1 + (\Delta/\delta\nu_H)^2]$$

The meaning of I_{sat} is discussed in class and is hopefully now clear; it is the intensity at which the initial population difference is reduced to half its value. If Δ is not zero, a higher intensity is needed to saturate the transition. Note also that the expression for I_{sat} can vary by an overall factor depending on the whether one is considering a 2-level system or a different multilevel system (for example, if the lower level is not the ground state for the atom).

(d) Make a plot of $a/(N\sigma_o)$ vs. I/I_{sat}^o for case $\Delta = 0$. We had previously seen that for a gas of two-level atoms in thermal equilibrium, a population inversion ($N_2 > N_1$) can not be achieved. By now, you should be convinced that under steady-state conditions, even a high-intensity laser can not invert a sample of two-level atoms. Thus a gas of two-level atoms is always an absorber.

(e) Finally, show that a can be written in the following form:

$$a = \left(\frac{\delta\nu_H}{\delta\nu'}\right)^2 \times \frac{a_o(\nu_o)}{1 + (\Delta/\delta\nu')^2},$$

where $\delta\nu' = \delta\nu_H\sqrt{1 + I/I_{sat}^o}$ is the *power broadened line width*. On the same graph, make a plot of $a/(N\sigma_o)$ versus detuning Δ for $I = 0, I = I_{sat}^o$, and $I = 3I_{sat}^o$.

4. Photon scattering. In this problem we will consider the same two-level system as in problem 2. In thermal equilibrium, with only blackbody radiation present, at any instant in time the vast majority of atoms in a gas will reside in the electronic ground state ψ_1. In steady-state equilibrium after near-resonant light has been turned on, each atom will then have a higher probability of being found in the excited state, due to interaction with the light.

The **rate** of photon scattering by a gas of atoms (i.e. the absorption and spontaneous emission of a photon from the beam of light) is an important quantity, and is useful to know in many applications, such as determining the amount of fluorescence observed as a laser beam passes through a gas of atoms. This scattering rate per atom (γ_{scat}) depends on the steady-state value for atoms in level 2 and the spontaneous emission rate from level 2:

$$\gamma_{scat} = A_2 \times \frac{N_2^{ss}}{N}.$$

(a) Using your result from problem 3(a) for N_2^{ss}, show that γ_{scat} can then be written as

$$\gamma_{scat} = \frac{A_2}{2} \cdot \left(\frac{s_0}{1+s_0}\right)\left(\frac{1}{1+\Delta^2/(\delta\nu_H')^2}\right),$$

where $s_0 = I/I_{sat}^o$ is the on-resonance saturation parameter, and $\delta\nu' = \delta\nu_H\sqrt{1+s_0}$.

In an experiment, you might imagine sending a tunable laser beam through a gas of atoms, collecting some of the fluorescence with a lens, and measuring the amount of this scattered light on a photodiode. As you scan the laser frequency through the atomic resonance, you'll see a power-broadened Lorentzian lineshape (as long as Doppler broadening can be neglected) with a full width at half maximum (FWHM) of $2 \times \delta\nu_{rad} = A_2/(2\pi)$. The peak scattering rate (when $\Delta = 0$) is given by $\gamma_{scat} = (A_2/2)\frac{s_0}{1+s_0}$, so the amount of scattered light will increase with I, but will saturate to a limiting value as I far exceeds I_{sat}.

(b) Calculate σ_0 for an atomic transition with a resonance at a wavelength of 780 nm.

(c) Calculate I_{sat}^o for an atomic transition with a FWHM linewidth of $A_2/(2\pi)$= 6 MHz .

(d) Suppose you let a confined gas of atoms interact with a resonant ($\Delta = 0$) laser beam with uniform intensity $I = I_{sat}$. Assume that 10^9 atoms are contained in a very tiny volume, approximately a single point in space. As the atoms interact with the light, they uniformly scatter photons into all directions at a rate γ_{scat}. A fraction δ of this light is collected by a lens of both diameter and focal length 5 cm, placed 5 cm away from the fluorescing atoms, and the collected light is focused onto a photodiode. If the atoms have a FHWM linewidth of 6 MHz, what is the power (in Watts) that is incident on the photodiode? [Detected Power = (fraction of total scattered light collected) x (energy per photon) x (number photons scattered per atom per unit time) x (number of atoms in the sample).]

Population inversion and gain in a 3-level system

When discussing gain rather than absorption, the population difference is defined as

$$\Delta N = N_2 - N_1,$$

and the gain coefficient is

$$g(\nu) = \Delta N \sigma(\nu).$$

The saturation of the gain is again due to the change in ΔN that results from stimulated transitions between levels 1 and 2. It is hopefully now clear from the previous problems that one cannot create a population inversion in a strictly 2-level system. The expression for ΔN depends on the particular multilevel system being studied. Most laser systems can be modeled by reducing them to an effective 3 or 4 level laser system. In this problem, we will assume there is no collisional or Doppler broadening.

5. (a) Write the rate equations for a 3-level atomic system as discussed in class, assuming the usual conditions and notation: (i) the lasing transition is between levels 1 and 2, where level 1 is the ground level, (ii) pumping is from level 1 to 3 at a rate per atom P, and (iii) instantaneous decay occurs from level 3 to 2.

(b) Solve the rate equations for steady state population densities N_1 and N_2. However, **do not** assume that field intensities are small. Keep the absorption and stimulated emission rate $R(I) = I \cdot \sigma/(\hbar\omega_o)$ in your equations. Be sure N_1 and N_2 are expressed only in terms of $P, R(I)$, A_{21}, and N, the total atom number density.

(c) Using your result above, solve for ΔN.

(d) Setting $R(I) \sim 0$, solve for ΔN^o and show that it matches the expression:

$$\Delta N^o = N \times \frac{P - A_{21}}{P + A_{21}}$$

(d) Finally, show that the population difference for a three level laser system can be written:

$$\Delta N = \frac{\Delta N^o}{1 + I/I_{sat}},$$

where the saturation intensity is now given by:

$$I_{sat} = \frac{\hbar\omega_0}{2\sigma} \times (A_{21} + P).$$

Note that if P=0, the gain coefficient is negative (absorption) and the saturation intensity is the same as previously expressed for the two level system.