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Astigmatic laser mode converters and transfer of orbital angular momentum

M.W. Beijersbergen, L. Allen, H.E.L.O. van der Veen and J.P. Woerdman
Huygens Laboratory, University of Leiden, P.O. Box 9504, 2300 RA Leiden, Netherlands

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We present the design of a mode converter which transforms a Hermite–gaussian mode of arbitrarily high order to a Laguerre–gaussian mode and vice versa. The converter consists of two cylindrical lenses and is based on appropriate use of the Gouy phase. We demonstrate mode conversion experimentally and consider where the concomitant transfer of orbital angular momentum is localized.

1. Introduction

A paraxial light beam of circular polarization is known to carry *spin* angular momentum of $+\hbar$ or $-\hbar$ per photon, for σ^+ or σ^- polarized beams, respectively. Interaction of such a light beam with a birefringent plate may lead to a mechanical torque, as was first demonstrated by Beth [1]. Recently we have shown theoretically that *orbital* angular momentum of light is also a useful concept for a paraxial light beam, particularly for a Laguerre–gaussian beam, which has a ($\exp\, \mathrm{i}l\phi$) azimuthal dependence [2]. Explicit calculation based on Maxwell's equations shows that such a beam, when linearly polarized, carries $l\hbar$ orbital angular momentum per photon. When the beam is circularly polarized, it carries $(l\pm1)\hbar$ as total angular momentum per photon. We have speculated that conversion of a paraxial beam with specific orbital angular momentum into another beam, with a different orbital angular momentum, will give rise to a torque on the converter. In the previous paper [2] the properties of the converter were only briefly alluded to. In this paper we discuss the design of the mode converter in detail, and report experimental demonstration of its ability to transform modes. We also analyze theoretically how the converter takes up the change in orbital angular momentum of the light beam.

The first part of our paper is, in fact, a generalization of previous work by Tamm on gaussian mode conversion [3,4]; we extend the results obtained by Tamm for low-order modes to modes of arbitrary order. There is also a strong connection with recent work by Abramochkin and Volostnikov [5]; they also deal with the influence of astigmatism on gaussian modes. They consider cases in which the astigmatism does not conserve the gaussian mode character of the incoming beam, in that the transverse intensity pattern of the outcoming beam changes upon propagation. We deal with suitable astigmatic elements which conserve the mode character and thus operate as mode *converters*; this restriction greatly simplifies the theoretical discussion.

2. Mode decomposition

In this section we introduce expansion formulas for a Hermite–gaussian (HG) and a Laguerre–gaussian (LG) mode which will turn out to be essential for an understanding of the mode converter. We use the following definitions for the amplitude of the Hermite–gaussian (HG) and Laguerre–gaussian (LG) laser modes which propagate along the z axis

$$u_{nm}^{\mathrm{HG}}(x,y,z)=C_{nm}^{\mathrm{HG}}(1/w)\exp[-\mathrm{i}k(x^2+y^2)/2R]$$
$$\times\exp[-(x^2+y^2)/w^2]\exp[-\mathrm{i}(n+m+1)\psi]$$
$$\times H_n(x\sqrt{2}/w)\,H_m(y\sqrt{2}/w)\,, \quad (1)$$

$$u_{nm}^{\mathrm{LG}}(r,\phi,z) = C_{nm}^{\mathrm{LG}}(1/w)\exp(-ikr^2/2R)\exp(-r^2/w^2) \times\exp[-i(n+m+1)\psi]\exp[-i(n-m)\phi] \times(-1)^{\min(n,m)}(r\sqrt{2}/w)^{|n-m|} \times L_{\min(n,m)}^{|n-m|}(2r^2/w^2)\,, \tag{2}$$

with

$$R(z)=(z_{\mathrm{R}}^2+z^2)/z\,, \tag{3}$$

$$\tfrac{1}{2}kw^2(z)=(z_{\mathrm{R}}^2+z^2)/z_{\mathrm{R}}\,, \tag{4}$$

$$\psi(z)=\arctan(z/z_{\mathrm{R}})\,. \tag{5}$$

$H_n(x)$ is the Hermite polynomial of order n, $L_p^l(x)$ is the generalized Laguerre polynomial [6], k is the wave number, and z_{R} is the Rayleigh range (half the confocal parameter) of the mode. We introduce $N=n+m$ as the order of the mode. Normalization of the amplitude such that $\int \mathrm{d}x\,\mathrm{d}y\,|u|^2=1$ yields

$$C_{nm}^{\mathrm{HG}}=\left(\frac{2}{\pi n!m!}\right)^{1/2}2^{-N/2}\,, \tag{6}$$

$$C_{nm}^{\mathrm{LG}}=\left(\frac{2}{\pi n!m!}\right)^{1/2}\min(n,m)\,. \tag{7}$$

Note that the indices we use for the LG mode differ from those normally used. The radial index p normally used is $\min(n,m)$, the minimum of n and m; the azimuthal index l is $n-m$. Our notation brings advantages in the context of the present paper: we shall show that a mode converter can transform a HG_{nm} mode into a LG_{nm} mode or vice versa.

By using relations between Hermite and Laguerre polynomials (see e.g. refs. [2–5]) one can show that a LG mode can be decomposed into a set of HG modes of the same order:

$$u_{nm}^{\mathrm{LG}}(x,y,z)=\sum_{k=0}^{N}\mathrm{i}^k\,b(n,m,k)\,u_{N-k,k}^{\mathrm{HG}}(x,y,z)\,, \tag{8}$$

with real coefficients

$$b(n,m,k)=\left(\frac{(N-k)!k!}{2^N n!m!}\right)^{1/2} \times\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}[(1-t)^n(1+t)^m]_{t=0}\,. \tag{9}$$

The factor i^k in eq. (8) corresponds to a $\pi/2$ relative phase difference between successive components. Perhaps surprisingly, a HG mode whose principal axes make an angle of 45° with the (x,y) axes (a 'diagonal' mode) can be decomposed, using relations between products of Hermite polynomials [2,5], into exactly the same constituent set:

$$u_{nm}^{\mathrm{HG}}\left(\frac{x+y}{\sqrt{2}},\frac{x-y}{\sqrt{2}},z\right) = \sum_{k=0}^{N} b(n,m,k)\,u_{N-k,k}^{\mathrm{HG}}(x,y,z)\,, \tag{10}$$

with the same real coefficients $b(n,m,k)$ as above. In this expansion, however, the successive components are in phase. In fig. 1 some examples of a mode decomposition of order 2 are given in diagrammatic form. In table 1 the coefficients $b(n,m,k)$ are given for modes up to order 3. For completeness we note that the relationship between the LG and HG modes can also be established via operator algebra [7] and by direct comparison with a 2D quantum harmonic oscillator [8].

3. Astigmatic Gouy phase

From eqs. (8),(10) it is clear that in order to perform the conversion from a HG mode to a LG mode one has to rephase the terms in the decomposition. This can be done by exploiting the Gouy phase $\psi(z)$

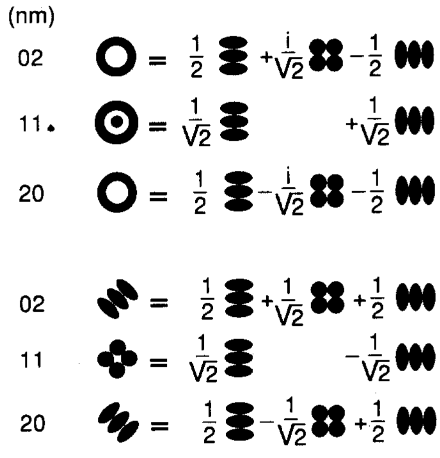

Fig. 1. Examples of the decomposition of HG_{nm} and LG_{nm} modes of order 2.

Table 1
The coefficients $b(n, m, k)$ which occur in eqs. (8), (10).

n	m	k=0	1	2	3
0	0	1			
0	1	$1/\sqrt{2}$	$1/\sqrt{2}$		
1	0	$1/\sqrt{2}$	$-1/\sqrt{2}$		
0	2	1/2	$1/\sqrt{2}$	1/2	
1	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	
2	0	1/2	$-1/\sqrt{2}$	1/2	
0	3	$1/\sqrt{8}$	$\sqrt{3/8}$	$\sqrt{3/8}$	$1/\sqrt{8}$
1	2	$\sqrt{3/8}$	$1/\sqrt{8}$	$-1/\sqrt{8}$	$-\sqrt{3/8}$
2	1	$\sqrt{3/8}$	$-1/\sqrt{8}$	$-1/\sqrt{8}$	$\sqrt{3/8}$
3	0	$1/\sqrt{8}$	$-\sqrt{3/8}$	$\sqrt{3/8}$	$-1/\sqrt{8}$

of a gaussian mode which appears in eqs. (1), (2), i.e. the phase shift that the beam undergoes when going through a waist as compared to that of a plane wave. Tamm has recognized that one can use the difference in Gouy phase between an astigmatic HG_{01} and HG_{10} mode to convert a first-order LG mode to a HG mode [3]. Here we show that an arrangement similar to that used by Tamm can transform modes of arbitrary order.

For an isotropic (i.e., non-astigmatic) gaussian beam the Gouy phase appears in eqs. (1), (2) as

$$(n+m+1)\,\psi(z)\,, \tag{11}$$

with $\psi(z)=\arctan(z/z_R)$ for a waist at position $z=0$. For an astigmatic beam the situation is different. Consider first an astigmatic HG beam which has its nodal lines parallel to the axes of the astigmatism. Such a beam can be produced by passing a HG beam through a cylindrical lens aligned along the axes of the mode pattern. The amplitude of this mode can be considered separately in the two transverse planes (x, z) and (y, z); in each plane the beam is characterized by the z-coordinate and the Rayleigh range of the waist. The resulting Gouy phase has two contributions, one from each transverse direction [9,10], and may be written as

$$(n+1/2)\,\psi_x(z)+(m+1/2)\,\psi_y(z)\,, \tag{12}$$

with

$$\psi_z=\arctan[(z-z_{0x})/z_{Rx}]\,, \tag{13}$$

$$\psi_y=\arctan[(z-z_{0y})/z_{Ry}]\,, \tag{14}$$

where z_{0x} and z_{0y} are the positions of the waists and z_{Rx} and z_{Ry} the corresponding Rayleigh ranges in the (x, z) and (y, z) planes, respectively. For an isotropic HG beam the two waists coincide and have equal diameters, so that eq. (12) reduces to eq. (11). It follows that isotropic HG beams of the same order $n+m$ have the same Gouy phase. But for the astigmatic HG beam ψ_x and ψ_y are different functions of z so that the relative phase of HG modes of the same order, but with different n and m, is a function of z.

It should be realized that neither a LG beam, nor a HG beam when passed through a cylindrical lens at an *arbitrary* angle relative to the mode pattern, can be described in the same way, since the amplitude of these beams is not separable in x and y. For such beams (and also for more general beams) one should decompose the transverse pattern into HG modes oriented along the axes of the lens. Since in an astigmatic beam the relative phase of these HG components changes upon propagation, the transverse pattern of these astigmatic beams will change accordingly. This applies to the cases considered in ref. [5].

4. Mode converter

In order to exploit the Gouy phase to construct a mode converter, the beam should be made astigmatic in a confined region only, while it is isotropic outside this region. When the beam is passed through this region, a definite phase difference will be introduced between the HG components which are oriented along the axes of astigmatism. Consider for example an astigmatic beam for which the waists coincide but have different Rayleigh ranges, z_{Rx} and z_{Ry}, respectively (fig. 2a). At the position where the two transverse radii of the astigmatic beam are equal, a cylindrical lens may be placed to match the radii of curvature of the beam such that the beam outside the lens is no longer astigmatic (fig. 2b). When the same cylindrical lens is also placed on the other side of the waist and if the input beam is properly mode-matched, the beam is astigmatic only between the two lenses (fig. 2c). The condition that the transverse radii of the beam (given by eq. (4)) are equal at the position of the lens $z=\pm d$ leads to

$$\frac{z_{Rx}^2+d^2}{z_{Rx}}=\frac{z_{Ry}^2+d^2}{z_{Ry}}, \tag{15}$$

and the condition that the input beam is mode-matched is fulfilled if the focal distance f of the cylindrical lenses satisfies (see eq. (3)),

$$\frac{1}{f}=\frac{1}{R_x(d)}-\frac{1}{R_y(d)}=\frac{d}{z_{Rx}^2+d^2}-\frac{d}{z_{Ry}^2+d^2}. \tag{16}$$

It is useful to introduce a parameter p such that

$$p=\sqrt{\frac{1-d/f}{1+d/f}}. \tag{17}$$

Equations (15), (16) then lead to

$$z_{Rx}=dp\,, \tag{18}$$

$$z_{Ry}=d/p\,. \tag{19}$$

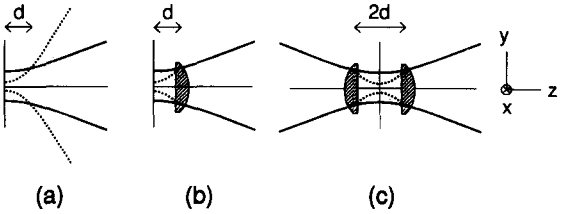

Fig. 2. Sketch of a symmetric mode converter. The dashed curve denotes the gaussian beam envelope in the (x, z) plane, and the solid curve that in the (y, z) plane. (a) An astigmatic waist at $z=0$. (b) A cylindrical lens matches the radii of curvature at $z=d$. (c) Two cylindrical lenses act as a converter on a mode-matched beam.

The change in Gouy phase $\psi(z)$ of a HG mode oriented along the axes of the lenses, when passed through this region, is, using eq. (12),

$$\Delta\psi=(n+m+1)(\Delta\psi_x+\Delta\psi_y)/2$$
$$+(n-m)(\Delta\psi_x-\Delta\psi_y)/2\,, \tag{20}$$

with

$$\Delta\psi_x=\psi_x(d)-\psi_x(-d)=2\arctan(d/z_{Rx}) \tag{21}$$

and analogously for $\Delta\psi_y$.

We now consider a 'diagonal' HG mode which is passed through the converter of fig. 2c and expand the input mode into HG modes of the same order $n+m$ oriented along the lens axes (eq. (10)). The successive terms in this expansion differ by two in the value of $(n-m)$. Therefore the total phase difference which is introduced *between* successive terms is

$$\theta=2[\arctan(d/z_{Rx})-\arctan(d/z_{Ry})]$$
$$=2[\arctan(1/p)-\arctan p]\,. \tag{22}$$

The phase difference is thus determined by the parameter p only and ranges from 0 to π.

If this phase difference is set equal to $\pi/2$ the system introduces a factor i^k in front of each term in the expansion of eq. (10), so that the HG mode is converted into a LG mode with the same indices n, m (eq. (8)). The condition $\theta=\pi/2$ is fulfilled if $p=-1+\sqrt{2}$, which leads to

$$d=f/\sqrt{2}\,. \tag{23}$$

Mode-matching (eq. (19)) requires that the input beam has a Rayleigh range (cf. fig. 2c)

$$z_{Ry}=f+d=(1+1/\sqrt{2})f. \tag{24}$$

We will call this converter, which converts a diagonal HG mode to a LG mode, a '$\pi/2$ converter', referring to the value of θ. Of course the argument can also be reversed: the $\pi/2$ relative phase different between the components of the LG mode can be removed with a $\pi/2$ converter, so that a HG mode is produced.

The converter with $\theta=\pi$, the 'π converter', implies $p=0$, and therefore

$$d=f,\quad z_{Rx}/f=0,\quad z_{Ry}/f=\infty\,, \tag{25}$$

which corresponds to a confocal configuration of the cylindrical lenses with a collimated incident beam. Note that the ideal π converter exists only in the geometrical optical limit. In practice, i.e. in a wave-optical description, z_{Rx} and z_{Ry} are always finite, so that $\theta=\pi-\epsilon$ instead of π. In a diffraction-limited system, ϵ can be made arbitrarily small by making the system (and incident-beam diameter) sufficiently large. In the geometrical optical limit, the lens system exchanges the left and the right side of the beam, so that a diagonal HG mode u_{nm}^{HG} is converted to u_{mn}^{HG}, and a LG mode u_{nm}^{LG} is converted to u_{mn}^{LG}, which has an azimuthal dependence of the opposite sign.

The $\pi/2$ and π converters are compared in fig. 3; here we have varied the distance between the cylindrical lenses, keeping their focal length constant. It illustrates that a $\pi/2$ converter generally requires a tightly focussed input beam, whereas a π converter operates on a collimated beam.

Obviously, such converters can also be constructed by using two cylindrical *mirrors*. Another possibility would be to exploit the astigmatism furnished by an off-axis configuration of spherical lenses and/or mirrors.

5. Experiments

For testing the mode converters described in sect. 4, we used a HeNe laser consisting of a gain tube of 35 cm length with Brewster windows (Spectra Physics 120S) in a two-mirror cavity (fig. 4). As we wanted this laser to operate in a higher-order transverse mode, we had to make the Fresnel number of the cavity as large as possible. Since the bore of the gain tube (∅ 1.8 mm) is the effective aperture in the cavity, this implies a small beam diameter at the Brewster windows. From eq. (4) it follows that the beam radius at a distance d from the waist has a minimum for $z_R=d$. Therefore we chose the position of the waist in the middle of the tube, with a Rayleigh range of about half the length of the tube ($z_R \approx 18$ cm). This was obtained with two commercial HeNe laser mirrors, one with 600 mm radius of curvature, placed at 525 mm from the middle of the gain tube, and the other with 437 mm radius of curvature, placed at 306 mm from the middle of the tube. The first mirror was a high-reflector for 633 nm, the second mirror was an output coupler with 1.2% transmission. In order to force the laser to operate in a higher-order Hermite–gaussian mode, two metal wires of 20 μm diameter were carefully positioned inside the cavity in front of the high-reflector, one vertical and the other horizontal, both perpendicular to the beam. The wires force the laser to operate in the higher-order HG mode which has nodal lines at the position of the wires. Any mode of order 0 up to 3 could be made this way, with output powers between 0.5 and 5 mW.

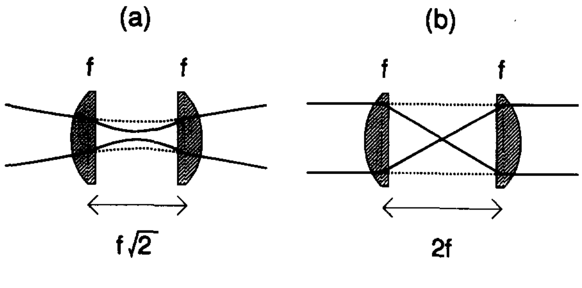

Fig. 3. Comparison of (a) a $\pi/2$ converter and (b) a π converter. Both converters consist of two identical cylindrical lenses of focal length f; they focus in the plane of the paper. The distance between the lenses is $f\sqrt{2}$ for the $\pi/2$ converter and $2f$ for the π converter. Dashed lines indicate the propagation of the beam in the other transverse direction.

We first tested the operation of the $\pi/2$ converter by mode-matching a 'diagonal' HG laser beam to such a converter by means of two spherical lenses. The cylindrical lenses of the converter had a focal distance of 19 mm, while the Rayleigh range of the input beam and the distance between the lenses were chosen according to eqs. (23), (24). A lens with a short focal length projected the output mode on to a screen, and this output pattern was photographed. The results shown in fig. 5 illustrate the conversion of several input HG modes to corresponding LG modes. When the $\pi/2$ converter was rotated around the propagation axis of the beam, an incoming HG beam was transformed to a HG mode when the converter was at 0 or 90 degrees and to a LG mode when the converter was at ± 45 degrees (fig. 6a). In another experiment the HG laser mode was converted into a LG mode with a fixed $\pi/2$ mode converter and then passed through a rotatable $\pi/2$ converter, which transformed the mode back to a HG mode. The output mode was photographed with the second converter at various angles (fig. 6b). The output is a HG

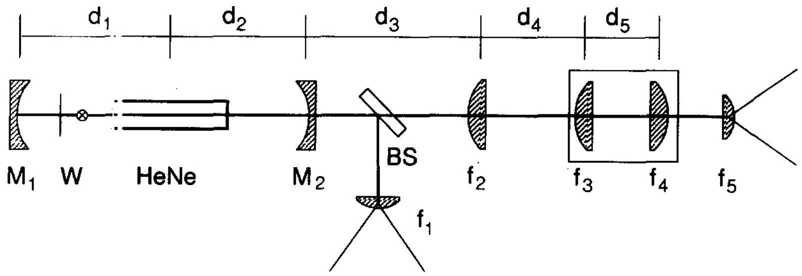

Fig. 4. Experimental arrangement used to demonstrate the operation of the $\pi/2$ mode converter. Two mutually perpendicular wires (W) are placed perpendicular to the mode axis inside a HeNe laser consisting of two mirrors (M_1, $R=600$ mm and M_2, $R=437$ mm) and a HeNe gain tube (HeNe). With a beam splitter (BS) part of the output is split off and projected onto a screen with a lens of short focal length (f_1). The beam is mode-matched with lens f_2 into a $\pi/2$ converter built with two cylindrical lenses (f_3 and f_4). The output is projected onto a screen with a lens of short focal length (f_5). Dimensions are: $d_1=525$ mm, $d_2=306$ mm, $d_3=225$ mm, $d_4=176$ mm, $d_5=27$ mm; $f_1=f_5=20$ mm, $f_2=160$ mm, $f_3=f_4=19$ mm (cylindrical).

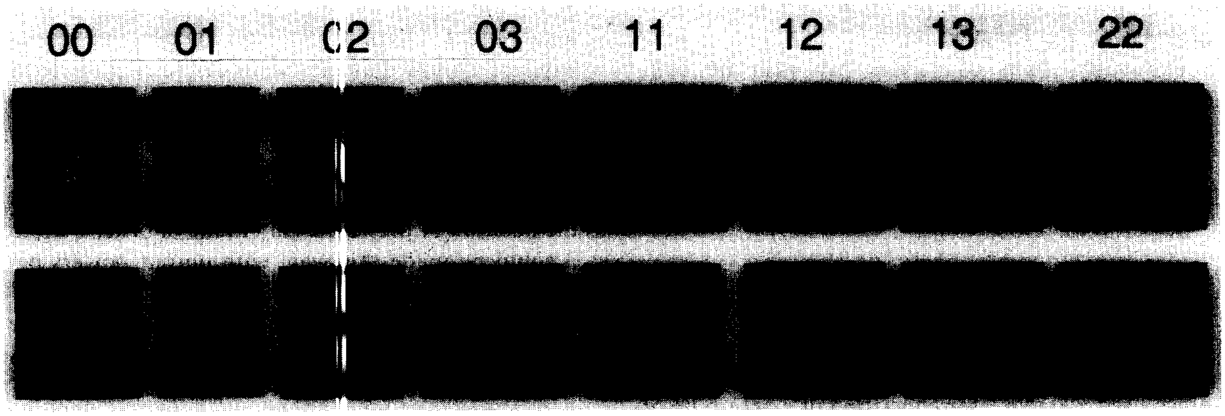

Fig. 5. Experimental results obtained with the $\pi/2$ converter. The top row shows the input HG_{nm} mode; the bottom the output LG_{nm} mode, where n, m is indicated above the modes.

mode which is always at 45 degrees with respect to the axes of the lenses of the converter.

To test the π converter we mounted such a converter in the rotatable mount and passed a collimated HG beam through it. The output is again a HG mode which rotates twice as fast as the converter, as shown in fig. 7.

To interpret these results it is helpful to keep in mind the resemblance between decomposition of a mode pattern on the one hand and decomposition of polarization on the other hand, as pointed out in ref. [2]. A quarter-wave plate converts linearly to circularly polarized light by introducing a $\pi/2$ phase difference between the linearly polarized components and is therefore analogous to the $\pi/2$ mode converter, which converts a HG mode into a LG mode by introducing a $\pi/2$ phase difference between the HG components. A half-wave plate converts left-handed to right-handed circularly polarized light by introducing a π phase difference and is therefore similar to the π converter.

Using this analogy, the experiment of fig. 6a is similar to passing a linearly polarized beam through a rotating quarter-wave plate. Depending on the orientation of the plate, this would result in a circularly or linearly polarized beam. The experiment of fig. 6b is similar to passing a circularly polarized beam through a rotating quarter-wave plate. In this case

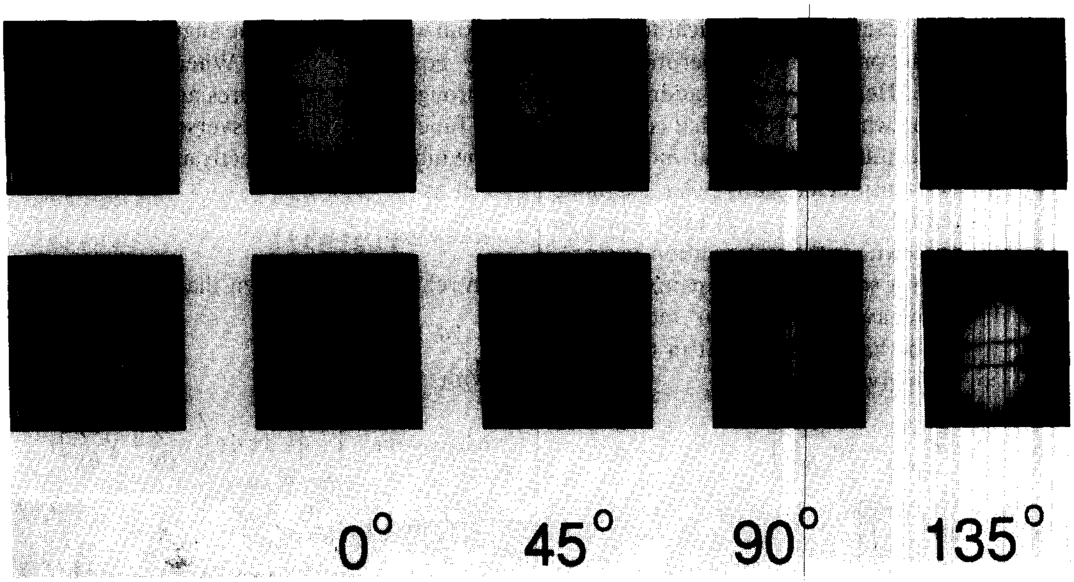

Fig. 6. Experimental results obtained with the $\pi/2$ converter at different angles, (a) with an input HG_{02} mode and (b) with an input LG_{02} mode. The left-most pattern in each row is the input, on the right are the output patterns with the angle of the converter indicated below.

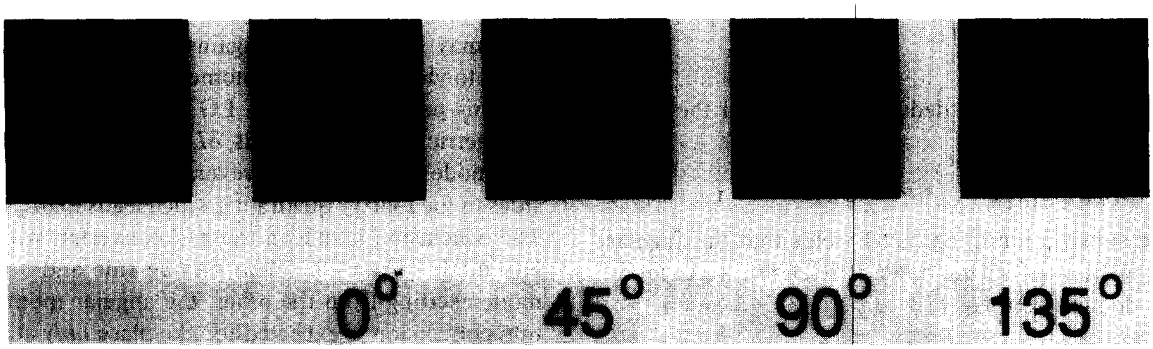

Fig. 7. Experimental results obtained with the π converter at different angles. On the left is the input HG_{02} mode, on the right the output patterns.

the output would be a linearly polarized beam whose polarization axis is at 45 degrees with respect to the axes of the plate. The last experiment (fig. 7) is similar to passing a linearly polarized beam through a half-wave plate. This would result in a linearly polarized beam whose polarization axis rotates twice as fast as the plate.

Note that in the paraxial approximation the polarization decomposition is two-dimensional, spanned by two polarization vectors. The mode decomposition has dimension $N+1$ for a mode of order N since such a mode can be expressed as a superposition of $N+1$ modes of the same order.

6. Angular momentum transfer

Now that we have studied in detail how a HG mode may be converted to a LG mode we remind the reader that it has been established previously that a LG beam carriers orbital angular momentum whereas a HG beam does not [2]. It may thus be expected that the transformation leads to a mechanical torque on the converter. We are preparing an experiment to investigate this issue. Here we wish to address theoretically the question where precisely, that is at which cylindrical lens, the transfer of angular momentum will take place. We consider a linearly polarized mode with vector potential $A = \boldsymbol{e}\, u(x, y, z) \exp(-ikz)$ in the Lorentz gauge (with $\boldsymbol{e}$ the polarization vector), where $u(x, y, z)$ is a solution of the paraxial wave equation. The time-averaged angular momentum density of this mode has a component in the direction of propagation given by [2]

$$M_z = i\omega\epsilon_0/2\left[\left(xu^* \frac{\partial u}{\partial y} - xu \frac{\partial u^*}{\partial y}\right) - \left(yu^* \frac{\partial u}{\partial x} - yu \frac{\partial u^*}{\partial x}\right)\right] = -\omega\epsilon_0 (\partial\alpha/\partial\phi)\,|u|^2 , \qquad (26)$$

with $\alpha = \arg u$. The angular momentum density per unit length in the direction of propagation is

$$L_z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, M_z . \qquad (27)$$

For a HG mode we deduce from eq. (1) the phase of u

$$\alpha = -kr^2/2R(z) - (n+m+1)\psi(z) \qquad (28)$$

and substitution in eq. (26) yields that the angular momentum of such a mode is zero. For a LG mode we deduce from eq. (2)

$$\alpha = -kr^2/2R(z) - (n+m+1)\psi(z) - (n-m)\phi , \qquad (29)$$

so that the ϕ derivative leads to an angular momentum proportional to $n-m$. It has been shown previously that it is in fact equal to $(n-m)\hbar$ per photon [2]. As the mode is linearly polarized, the intrinsic spin angular momentum is zero and the angular momentum found here is orbital angular momentum.

The $\pi/2$ converter transforms a HG mode without orbital angular momentum to a LG mode with orbital angular momentum. It might be anticipated that the transfer of angular momentum takes place at both lenses, but it turns out that this is not the case. For any mode $u(x, y, z)$ which is a solution of the paraxial wave equation, the angular momentum is given by eqs. (26), (27). When this mode is passed through a lens it acquires an extra phase χ which is a function of the transverse coordinates [#1], so that the mode function directly after the lens can be written as

$$u' = u \exp[i\chi(x, y)] . \qquad (30)$$

In ref. [11] it is shown that

$$L_z(u') = L_z(u) + \delta L_z , \qquad (31)$$

with

$$\delta L_z = -\omega\epsilon_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy \left(x\frac{\partial\chi}{\partial y} - y\frac{\partial\chi}{\partial x}\right) \times |u(x, y, z)|^2 . \qquad (32)$$

For a cylindrical lens we have $\chi = -\frac{1}{2}(k/f)x^2$, so that

$$\delta L_z = -\omega\epsilon_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, (k/f)xy\, |u(x, y, z)|^2 . \qquad (33)$$

This may be normalized against the energy in the beam to yield the angular momentum transferred to the lens per photon. For a LG mode, as $|u_{nm}^{LG}|^2$ is symmetric around the z axis, δL_z is zero. A diagonal HG mode with $m \neq n$, however, has unequal intensities in the four xy quadrants, which leads to $\delta L_z \neq 0$. The conclusion is that in the $\pi/2$ converter, with a HG mode $(m \neq n)$ incident on one side and a LG mode exciting from the other, the angular momentum conversion is expected to take place *only* at the lens which sees the input HG mode. It follows that the astigmatic HG mode just after the first lens already contains the orbital angular momentum of the output LG mode. The function of the second lens is to change the astigmatic beam into a pure LG mode. For the same reason, if a LG mode is incident on a $\pi/2$ converter and is converted into a HG mode, the transfer of orbital angular momentum is expected to take place at the second lens.

In fact, for a macroscopic lossless dielectric object such as a lens the radiative force is a conservative force ("gradient force") which is derived from the

[#1] Writing the operation of the lens as a phase factor is only correct for a lens whose principal planes coincide, i.e. a thin lens.

dielectric polarization energy as the potential [12]. Dielectric matter is drawn to positions where the electric field strength is largest (as an example, when the plates of a capacitor are dipped into a dielectric fluid, the fluid is drawn into the volume between the plates). This explains why the transfer of angular momentum, that is the mechanical torque, occurs at the 'HG side' of the $\pi/2$ converter only. For a cylindrical lens in a HG beam the dielectric polarization energy clearly depends on the angle between the cylinder axis and the HG axes x and y. The cylindrical lens will tend to align with the maximum optical intensity inside the dielectric. A radiative torque on the lens results unless the polarization energy is minimum or maximum, that is unless the cylinder axis is aligned with x or y. For a cylinder lens in a LG beam the radiative torque is zero as the polarization energy is independent of its orientation.

It follows from eq. (33), and also from the energy arguments above, that the change in orbital angular momentum may in principle be made arbitrarily high by choosing the lens arbitrarily strong ($f \to 0$). It follows similarly that the orbital angular momentum may be changed by astigmatic elements other than a cylinder lens (e.g. a prism). Also, for similar reasons, a LG mode with an angular sector removed with a suitable mask *will* transfer orbital angular momentum to a cylindrical lens. However, all such operations will not lead to a gaussian mode at the output and thus do not correspond to a well-defined mode conversion.

The π converter seems an anomaly in this context. If the perfect π converter has a LG mode incident at the first lens, a LG mode of opposite handedness will leave the second lens. According to the arguments given above it appears that there is no orbital angular momentum exchange on either of the lenses, although the orbital angular momentum of the beam changes sign. As noted in sect. 4, however, a perfect π converter does not exist. If a perfect LG mode is incident on a practical converter, the output mode will always, however slightly, differ from a perfect LG mode, which invalidates the analysis give above. An appropriate analysis of this converter has been given by Van Enk and Nienhuis [11].

7. Conclusions

We have shown that it is possible to build a mode converter with two cylindrical lenses which converts a Hermite–gaussian mode or arbitrary order into a Laguerre–gaussian mode of the same order and vice versa. The conversion is described using a mode analysis, based on the decomposition of a LG mode and a diagonally oriented HG mode into Hermite–gaussians. Two converters are introduced: the $\pi/2$ converter which converts a HG to a LG mode or vice versa, and the π converter which exchanges the indices of the incoming mode and thereby converts a LG mode into one with opposite azimuthal dependence. We have demonstrated these conversions experimentally and have shown the analogy between the mode converters and half- and quarter-wave plates. The transfer of orbital angular momentum, occurring when a HG mode without orbital angular momentum is converted into a LG mode with orbital angular momentum, is shown to take place at the lens onto which the HG mode is incident.

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