Problem 4) a) Each voltage and current in the circuit has a Fourier transform; that is,

$$V_S(s) = \int_{-\infty}^{\infty} v_S(t) e^{-\mathrm{i}2\pi st}\mathrm{d}t, \tag{1}$$

$$V_C(s) = \int_{-\infty}^{\infty} v_C(t) e^{-\mathrm{i}2\pi st}\mathrm{d}t, \tag{2}$$

$$V_L(s) = \int_{-\infty}^{\infty} v_L(t) e^{-\mathrm{i}2\pi st}\mathrm{d}t, \tag{3}$$

$$I(s) = \int_{-\infty}^{\infty} i(t) e^{-\mathrm{i}2\pi st}\mathrm{d}t, \tag{4}$$

$$I_{1,2}(s) = \int_{-\infty}^{\infty} i_{1,2}(t) e^{-\mathrm{i}2\pi st}\mathrm{d}t. \tag{5}$$

The time-dependent voltages and currents may be written as inverse Fourier transforms, namely,

$$v_C(t) = \int_{-\infty}^{\infty} V_C(s) e^{\mathrm{i}2\pi st}\mathrm{d}s, \tag{6}$$

$$v_L(t) = \int_{-\infty}^{\infty} V_L(s) e^{\mathrm{i}2\pi st}\mathrm{d}s, \tag{7}$$

$$i_{1,2}(t) = \int_{-\infty}^{\infty} I_{1,2}(s) e^{\mathrm{i}2\pi st}\mathrm{d}s. \tag{8}$$

In the Fourier domain, the capacitor's current and voltage have a simple proportionality relation:

$$i_1(t) = C\,\mathrm{d}v_C(t)/\mathrm{d}t = C\int_{-\infty}^{\infty}(\mathrm{i}2\pi s)V_C(s)e^{\mathrm{i}2\pi st}\mathrm{d}s \quad \rightarrow \quad I_1(s) = \mathrm{i}2\pi CsV_C(s). \tag{9}$$

Similarly, the inductor's current and voltage have a simple proportionality relation, as follows:

$$v_L(t) = L\,\mathrm{d}i_2(t)/\mathrm{d}t = L\int_{-\infty}^{\infty}(\mathrm{i}2\pi s)I_2(s)e^{\mathrm{i}2\pi st}\mathrm{d}s \quad \rightarrow \quad V_L(s) = \mathrm{i}2\pi LsI_2(s). \tag{10}$$

Since the voltage drop across the R_1C branch equals the voltage drop across the R_2L branch, upon substitution for $I_1(s)$ from Eq.(9) and for $V_L(s)$ from Eq.(10), we find

$$V_C + R_1I_1 = R_2I_2 + V_L \;\rightarrow\; (1 + \mathrm{i}2\pi R_1Cs)V_C = (R_2 + \mathrm{i}2\pi Ls)I_2 \;\rightarrow\; I_2(s) = \left(\frac{1+\mathrm{i}2\pi R_1Cs}{R_2+\mathrm{i}2\pi Ls}\right)V_C(s). \tag{11}$$

The source voltage $V_S(s)$ drops across the resistor R_0 and the R_1C branch, yielding

$$V_S = R_0(I_1 + I_2) + V_C + R_1I_1. \tag{12}$$

b) Substituting into Eq.(12) for $I_1(s)$ and $I_2(s)$ from Eqs.(9) and (11), we finally arrive at

$$V_S = \mathrm{i}2\pi(R_0 + R_1)CsV_C + R_0\left(\frac{1 + \mathrm{i}2\pi R_1Cs}{R_2 + \mathrm{i}2\pi Ls}\right)V_C + V_C. \tag{13}$$

Thus, in the Fourier domain, the capacitor voltage is proportional to the source voltage; that is,

$$V_C(s) = -\frac{R_2 + \mathrm{i}2\pi Ls}{4\pi^2(R_0+R_1)LCs^2 - \mathrm{i}2\pi[L+(R_0R_1+R_0R_2+R_1R_2)C]s - (R_0+R_2)}V_S(s) = H(s)V_S(s). \tag{14}$$

The transfer function $H(s)$ is seen to be the ratio of a first-order polynomial (in s) to a second-order polynomial (in s).

c) For an impulsive voltage source, $V_S(s) = \mathcal{F}\{\delta(t)\} = 1$, Eq.(14) yields $V_C(s) = H(s)$. Thus, the transfer function is the Fourier transform of the impulse-response $h(t) = \int_{-\infty}^{\infty} H(s)e^{\mathrm{i}2\pi st}\mathrm{d}s$. The capacitor's impulse-response, $v_C(t) = h(t)$, behaving as a damped harmonic oscillator, can be computed by integration in the complex s-plane over infinitely large semi-circular contours.

Since both poles s_1 and s_2 of the transfer function are in the s-plane's upper-half, the integral over a semi-circular contour closing in the lower-half s-plane yields $v_C(t) = 0$ for $t < 0$. As for $t > 0$, the integral must be computed over a large semi-circular loop closing in the upper-half of the s-plane. The sum of the residues at the two poles (s_1 and s_2) yields the capacitor's voltage $v_C(t)$ for $t > 0$. While the real part of the poles corresponds to the oscillation frequency, their imaginary part represents the time constant of the exponential decay caused by the resistive losses in R_0, R_1, and R_2.

Digression. The denominator of $V_C(s)$, which, for the impulsive voltage source, equals $H(s)$, has two roots, namely,

$$\begin{aligned} s_{1,2} &= \mathrm{i}\pi[L + (R_0R_1 + R_0R_2 + R_1R_2)C] \\ &\qquad \pm\sqrt{-\pi^2[L + (R_0R_1 + R_0R_2 + R_1R_2)C]^2 + 4\pi^2(R_0 + R_1)(R_0 + R_2)LC} \\ &= \mathrm{i}\pi[L + (R_0R_1 + R_0R_2 + R_1R_2)C] \pm \pi\sqrt{4R_0^2LC - [L - (R_0R_1 + R_0R_2 + R_1R_2)C]^2}. \end{aligned} \tag{15}$$

If the expression under the radical is positive, both s_1 and s_2 will lie in the upper-half of the complex s-plane — one in the first quadrant and the other in the second. In this case, the real parts of s_1 and s_2 determine the oscillation frequency of $v_C(t)$, while the imaginary part dictates the exponential decay rate. In contrast, if the square root turns out to be imaginary, then s_1 and s_2 will be exclusively on the s-plane's imaginary axis, resulting in purely exponential decay of $v_C(t)$ without oscillation. An outline of the argument showing that both imaginary roots in the latter case are in the upper-half of the s-plane is provided below.

$$\begin{aligned} &[L + (R_0R_1 + R_0R_2 + R_1R_2)C] \pm \sqrt{[L - (R_0R_1 + R_0R_2 + R_1R_2)C]^2 - 4R_0^2LC} \geq 0 \\ \rightarrow\quad &[L + (R_0R_1 + R_0R_2 + R_1R_2)C]^2 \geq [L - (R_0R_1 + R_0R_2 + R_1R_2)C]^2 - 4R_0^2LC \\ \rightarrow\quad &4(R_0R_1 + R_0R_2 + R_1R_2)LC + 4R_0^2LC \geq 0 \quad \rightarrow \quad 4(R_0 + R_1)(R_0 + R_2)LC \geq 0. \end{aligned} \tag{16}$$