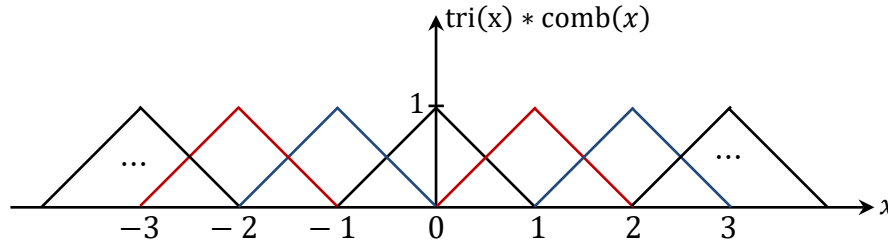
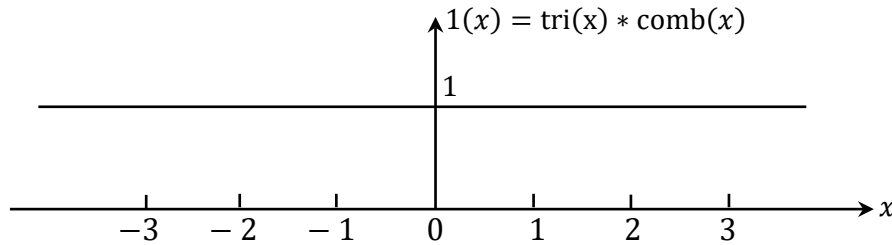


**Problem 3** a) Upon convolution, each  $\delta$ -function comprising the comb function shifts the triangle to the location of the  $\delta$ -function. The end result is a superposition of shifted triangular functions shown in the figure below.



In this figure, the overlapping positive-slope and negative-slope edges of adjacent triangles add up to 1.0, resulting in the unit constant function  $1(x)$  depicted below.



b) The Fourier transform of  $1(x)$  can be computed in several alternative ways. For instance, let  $1(x) = \lim_{L \rightarrow \infty} \text{rect}(x/L)$ . Then, by the scaling theorem of Fourier transformation,  $\mathcal{F}\{1(x)\} = \lim_{L \rightarrow \infty} L \text{sinc}(Ls)$ . This narrow, tall, symmetric sinc function has an area equal to 1. This is because the area under the Fourier transform of a function equals the value of the function at  $x = 0$ , which, in the present case, is  $\text{rect}(0) = 1$ . Consequently,  $\mathcal{F}\{1(x)\} \rightarrow \delta(s)$ .

Alternatively, let  $1(x) = \lim_{\epsilon \rightarrow 0} e^{-(\epsilon x)^2}$ . Considering that  $\mathcal{F}\{e^{-\pi x^2}\} = e^{-\pi s^2}$ , the scaling theorem yields

$$\mathcal{F}\{e^{-(\epsilon x)^2}\} = \mathcal{F}\{e^{-\pi(\epsilon x/\sqrt{\pi})^2}\} = (\sqrt{\pi}/\epsilon)e^{-\pi(\sqrt{\pi}s/\epsilon)^2} = (\sqrt{\pi}/\epsilon)e^{-\pi^2(s/\epsilon)^2}.$$

This narrow, tall, symmetric Gaussian has an area equal to 1. (The area under the function is unity, as the integral of a transform equals the value of the function at  $x = 0$ , which is  $e^0 = 1$  in the present case.) Consequently,  $\mathcal{F}\{1(x)\} = \lim_{\epsilon \rightarrow 0} \{(\sqrt{\pi}/\epsilon)e^{-\pi^2(s/\epsilon)^2}\} = \delta(s)$ .

As an alternative third method, consider the function  $f(x) = e^{-\epsilon|x|}$ , which approaches  $1(x)$  in the limit of  $\epsilon \rightarrow 0$ . The Fourier transform of this function is

$$\begin{aligned} F(s) &= \int_{-\infty}^0 e^{\epsilon x} e^{-i2\pi s x} dx + \int_0^{\infty} e^{-\epsilon x} e^{-i2\pi s x} dx = \frac{e^{(\epsilon-i2\pi s)x}}{\epsilon-i2\pi s} \Big|_{x=-\infty}^0 - \frac{e^{-(\epsilon+i2\pi s)x}}{\epsilon+i2\pi s} \Big|_{x=0}^{\infty} \\ &= \frac{1}{\epsilon-i2\pi s} + \frac{1}{\epsilon+i2\pi s} = \frac{2\epsilon}{\epsilon^2+4\pi^2s^2} = \frac{2/\epsilon}{1+(2\pi s/\epsilon)^2}. \end{aligned}$$

It is seen that  $F(s)$  is a tall, narrow, symmetric function whose area equals 1.0, since its inverse Fourier transform,  $f(x) = e^{-\varepsilon|x|}$ , equals 1.0 at  $x = 0$ . In the limit when  $\varepsilon \rightarrow 0$ , the function  $F(s)$  approaches  $\delta(s)$ .

c) The Fourier transform of the convolution  $\text{tri}(x) * \text{comb}(x)$  is the product  $\text{sinc}^2(s)\text{comb}(s)$ . Considering that  $\text{sinc}(s) = 0$  at  $s = \pm 1, \pm 2, \pm 3, \dots$ , all  $\delta$ -functions comprising  $\text{comb}(s)$ , when multiplied by zero, are nullified. The only surviving  $\delta$ -function is the one at  $s = 0$ , where  $\text{sinc}^2(0) = 1$ . Consequently,  $\text{sinc}^2(s)\text{comb}(s) = \delta(s)$ , which is the Fourier transform of  $1(x)$ .

---