**Opti 403A/503A** **Final Exam** (5/11/2026) **Time: 2 hours**

**Please write your name and ID number on all the pages, then staple them together.**
**Answer all the questions.**

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4 pts **Problem 1**) Let the function $f(z)$ of the complex variable $z = x + \mathrm{i}y$ contain both $z$ and its complex conjugate $z^*$. Typical examples are $f(z) = |z|^2 = zz^*$ and $f(z) = |z|^4 = z^2 z^{*2}$. One may choose to overlook the complex nature of the variable $z$ by treating $f(\cdot)$ as a function of the real-valued variables $x$ and $y$. With this caveat, show that the partial derivatives of $f(x, y)$ with respect to $x$ and $y$ can be expressed as

$$\partial f/\partial x = \partial f/\partial z + \partial f/\partial z^*,$$

$$\partial f/\partial y = \mathrm{i}(\partial f/\partial z - \partial f/\partial z^*).$$

Confirm the validity of the above results in the following cases:

2 pts i) $f(x, y) = |z|^2 = x^2 + y^2$.

2 pts ii) $f(x, y) = |z|^4 = x^4 + 2x^2y^2 + y^4$.

4 pts iii) $f(x, y) = e^{|z|^2 + z} = e^{x^2 + y^2 + x + \mathrm{i}y}$.

**Hint**: Given the functions $g(x, y)$ and $h(x, y)$, the multivariate chain rule of differentiation for the composite function $f(g, h)$ states that $\partial f/\partial x = (\partial f/\partial g)(\partial g/\partial x) + (\partial f/\partial h)(\partial h/\partial x)$, with a corresponding expression for $\partial f/\partial y$.

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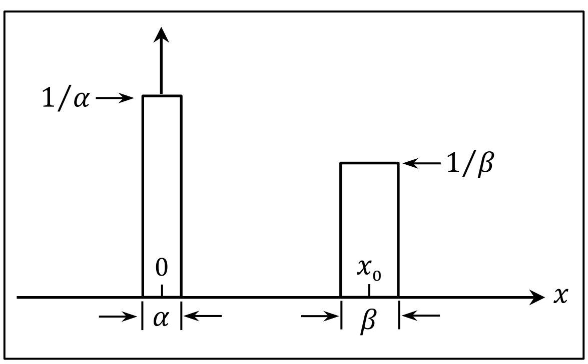
12 pts **Problem 2**) The goal of this problem is to demonstrate that $\delta(x) * \delta(x - x_0) = \delta(x - x_0)$. For sufficiently small values of $\alpha$ and $\beta$, suppose $\delta(x)$ is defined as $\alpha^{-1}\mathrm{rect}(x/\alpha)$, and $\delta(x - x_0)$ as $\beta^{-1}\mathrm{rect}[(x - x_0)/\beta]$. It is not necessary for $\alpha$ and $\beta$ to be equal; you may assume that $\beta > \alpha$, as depicted in the figure.




To this pair of rectangular functions, apply the standard procedure of convolution operation, namely, "*flip* → *shift* → *multiply* → *integrate* → *repeat*," then confirm that the resulting convolution converges to $\delta(x - x_0)$ in the limit of $\alpha \to 0$ and $\beta \to 0$.

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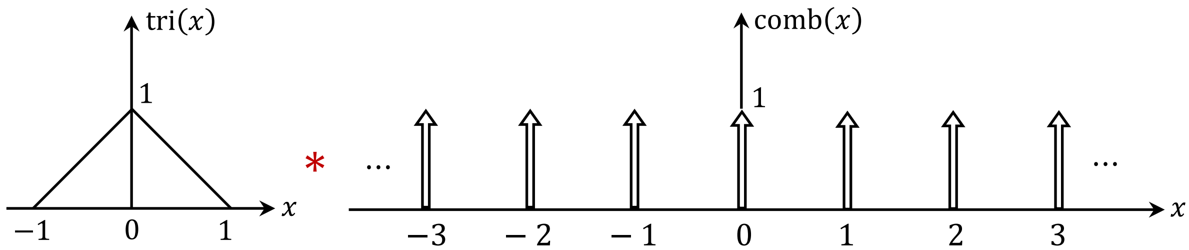
**Problem 3**) The Fourier transform of the triangular function, defined as $\text{tri}(x) = 1 - |x|$ when $|x| \le 1$, and $\text{tri}(x) = 0$ when $|x| > 1$, is $\text{sinc}^2(s) = [\sin(\pi s)/(\pi s)]^2$. The Fourier transform of the function $\text{comb}(x)$ is $\text{comb}(s)$.




5 pts a) Plot the function $\text{tri}(x) * \text{comb}(x)$, the convolution of the triangular function with the comb function. Based on the visual appearance of the plot, argue that the resulting function is $1(x)$, i.e., a function that equals 1.0 for all values of $x$ from $-\infty$ to $\infty$.

4 pts b) Show that the Fourier transform of $1(x)$ is $\delta(s)$. This can be done, for instance, by treating $1(x)$ as the limiting form of $\text{rect}(x/L)$ when $L \to \infty$, or as the limit of $e^{-\varepsilon|x|}$ when $\varepsilon \to 0$.

4 pts c) Justify the result obtained in part (a) by examining the transforms of the three functions, namely, $\text{sinc}^2(s)$, $\text{comb}(s)$, and $\delta(s)$, in the Fourier domain.

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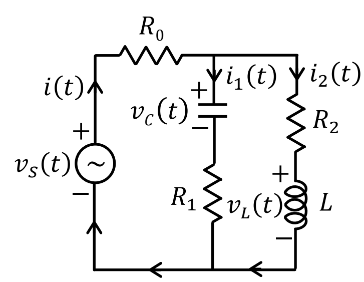
**Problem 4**) An electric circuit consists of two parallel branches, one with the capacitor $C$ in series with the resistor $R_1$, the other with $R_2$ in series with the inductor $L$. The parallel branches are driven by a voltage source, $v_s(t)$, in series with the resistor $R_0$.




5 pts a) Define Fourier-transformed functions $V(s) = \int_{-\infty}^{\infty} v(t)e^{-i2\pi st}\mathrm{d}t$ and $I(s) = \int_{-\infty}^{\infty} i(t)e^{-i2\pi st}\mathrm{d}t$ for each element and each branch, then apply Kirchhoff's current and voltage laws to relate them. (Use the differentiation theorem of Fourier transformation to replace time-derivatives with algebraic expressions.)

4 pts b) Solve the Kirchhoff equations for the capacitor voltage $V_C(s)$, relating it to the source voltage $V_S(s)$ via a transfer function $H(s)$; that is, $V_C(s) = H(s)V_S(s)$. (The transfer function turns out to be a first-order polynomial in $s$, divided by a second-order polynomial in $s$, with constant coefficients that are combinations of $R_0, R_1, R_2, C$, and $L$.)

4 pts c) Assuming an impulsive voltage source, $v_s(t) = \delta(t)$, determine the (Fourier-transformed) capacitor voltage $V_C(s)$. Explain the general characteristics of $v_C(t)$ qualitatively, without performing a full complex contour integration for the inverse transform.

**Hint**: For resistors: $v(t) = Ri(t)$; for capacitors: $i(t) = C\,\mathrm{d}v(t)/\mathrm{d}t$; for inductors: $v(t) = L\,\mathrm{d}i(t)/\mathrm{d}t$.

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