Problem 3) The perpendicular to $\overline{AC}$ is along the direction of $A\mathrm{i} = (a + \mathrm{i}c)\mathrm{i}$. Scaling this by a real-valued constant s_1 (to be determined) and positioning it at $B = a + b$ yields the orthocenter (i.e., the crossing point of the altitudes) at $B + s_1 A\mathrm{i} = (a + b) + s_1(a + \mathrm{i}c)\mathrm{i} = (a + b - s_1 c) + \mathrm{i}s_1 a$. The perpendicular to $\overline{AB}$ is $(A - B)\mathrm{i} = [(a + \mathrm{i}c) - (a + b)]\mathrm{i} = -(c + \mathrm{i}b)$. Scaling this by another real-valued constant s_2 (to be determined) and positioning it at the origin (i.e., at $C = 0$) yields the crossing point at $-s_2(c + \mathrm{i}b)$. Equating these two points, we find

$$(a + b - s_1 c) + \mathrm{i}s_1 a = -s_2 c - \mathrm{i}s_2 b \quad \rightarrow \quad a + b - s_1 c = -s_2 c \quad \text{and} \quad s_1 a = -s_2 b$$

$$\rightarrow \quad s_1 = b/c, \quad s_2 = -a/c.$$

Thus, the crossing point is at $-s_2(c + \mathrm{i}b) = a + \mathrm{i}(ab/c)$. Clearly, this point is located on the third altitude, which is the perpendicular dropped from A onto $\overline{BC}$, as depicted in the figure. It also shows that the crossing point is at a distance of ab/c above the base $\overline{BC}$.

Digression. A similar procedure can be used to demonstrate that the three medians cross at a single point (the triangle's centroid). The median of $\overline{AC}$ is

$$B - \tfrac{1}{2}A = (a + b) - \tfrac{1}{2}(a + \mathrm{i}c) = (\tfrac{1}{2}a + b) - \tfrac{1}{2}\mathrm{i}c.$$

Scaling this by the (as yet unknown) real-valued coefficient s_1 and positioning it at B yields the crossing point (located somewhere along $\overline{AC}$), as follows:

$$(a + b) + s_1[(\tfrac{1}{2}a + b) - \tfrac{1}{2}\mathrm{i}c].$$

Similarly, the median of $\overline{AB}$ is $\tfrac{1}{2}(A + B) = a + \tfrac{1}{2}(b + \mathrm{i}c)$, which must be scaled by the (as yet unknown) real-valued coefficient s_2, then equated with the presumed crossing point on the median of $\overline{AC}$. We will have

$$(a + b) + s_1[(\tfrac{1}{2}a + b) - \tfrac{1}{2}\mathrm{i}c] = s_2[a + \tfrac{1}{2}(b + \mathrm{i}c)]$$

$$\rightarrow \quad a + b + \tfrac{1}{2}s_1 a + s_1 b = s_2 a + \tfrac{1}{2}s_2 b \quad \text{and} \quad s_1 = -s_2$$

$$\rightarrow \quad (a + b)(1 + \tfrac{3}{2}s_1) = 0 \quad \rightarrow \quad s_1 = -s_2 = -\tfrac{2}{3}.$$

Since the distance from each of the vertices B and C to the crossing point is ⅔ of the length of the corresponding median, the third median (i.e., that from A to $\overline{BC}$) is obligated to pass through the same crossing point. The triangle's centroid is thus located at

$$s_2[a + \tfrac{1}{2}(b + \mathrm{i}c)] = \tfrac{1}{3}(2a + b) + \tfrac{1}{3}\mathrm{i}c.$$

As for the triangle's circumcenter (i.e., crossing point of its three perpendicular bisectors), we follow similar lines of reasoning to write

$$\perp \text{ bisector of } \overline{AC}: \quad \tfrac{1}{2}A + s_1 A\mathrm{i} = \tfrac{1}{2}(a + \mathrm{i}c) + s_1(\mathrm{i}a - c) = (\tfrac{1}{2}a - s_1 c) + \mathrm{i}(\tfrac{1}{2}c + s_1 a),$$

$$\perp \text{ bisector of } \overline{AB}: \quad \tfrac{1}{2}(A + B) + s_2(A - B)\mathrm{i} = \tfrac{1}{2}(a + \mathrm{i}c + a + b) + s_2(a + \mathrm{i}c - a - b)\mathrm{i}$$

$$= (a + \tfrac{1}{2}b - s_2 c) + \mathrm{i}(\tfrac{1}{2}c - s_2 b).$$

Equating the above crossing points (located on the ⊥ bisectors of $\overline{AC}$ and $\overline{AB}$, respectively), we arrive at

$$(\tfrac{1}{2}a - s_1 c) + \mathrm{i}(\tfrac{1}{2}c + s_1 a) = (a + \tfrac{1}{2}b - s_2 c) + \mathrm{i}(\tfrac{1}{2}c - s_2 b)$$

$$\rightarrow \quad s_2 - s_1 = (a+b)/2c \qquad \text{and} \qquad s_2 = -(a/b)s_1$$

$$\rightarrow \quad s_1 = -b/2c \qquad \text{and} \qquad s_2 = \frac{a}{2c}.$$

Thus, the ⊥ bisectors of $\overline{AB}$ and $\overline{AC}$ cross at

$$(\tfrac{1}{2}a - s_1 c) + \mathrm{i}(\tfrac{1}{2}c + s_1 a) = \tfrac{1}{2}(a+b) + \tfrac{1}{2}\mathrm{i}(c^2 - ab)/c.$$

The crossing point is located on the ⊥ bisector of $\overline{BC}$, since its real part is $\tfrac{1}{2}(a+b)$.

The three crossing points discussed above are collinear (i.e., located on a straight line known as the triangle's Euler line). To see this, consider the points z_1, z_2, z_3 in the complex plane. For these points to be collinear, the numbers $z_2 - z_1$ and $z_3 - z_1$ must share the same orientation in the complex plane; that is, $z_3 - z_1 = s(z_2 - z_1)$ for some real factor s. This rearranges to $(s-1)z_1 - sz_2 + z_3 = 0$, demonstrating that z_1, z_2, z_3 are linearly dependent with real, nonzero coefficients that sum to zero. As seen below, the orthocenter, the centroid, and the circumcenter of our ABC triangle exhibit such a linear dependency and are, therefore, collinear.

$$\tfrac{3}{2}[\tfrac{1}{3}(2a+b) \underset{\substack{\uparrow \\ \boxed{\text{centroid}}}}{+} \tfrac{1}{3}\mathrm{i}c] - \tfrac{1}{2}[a \underset{\substack{\uparrow \\ \boxed{\text{orthocenter}}}}{+} \mathrm{i}(ab/c)] = \tfrac{1}{2}(a+b) \underset{\substack{\uparrow \\ \boxed{\text{circumcenter}}}}{+} \tfrac{1}{2}\mathrm{i}(c^2 - ab)/c.$$

Generally, a necessary condition for $z_1, z_2, \cdots, z_n$ to be collinear in the complex plane is the existence of real, nonzero coefficients $c_1, c_2, \cdots, c_n$ such that $\sum_{k=1}^{n} c_k = 0$ and $\sum_{k=1}^{n} c_k z_k = 0$. While this holds for any set of distinct points on a single line, the condition is necessary but *not* sufficient. For example, if $\{z_1, z_2, z_3\}$ and $\{z_4, z_5, z_6\}$ form two distinct lines, the sum of their respective linear-dependency equations satisfies the criteria for the combined six-point set without requiring all six points to be mutually collinear. As for the proof of necessity, it has already been given for $n = 3$, which can be extended to $n > 3$ by induction. For instance, to incorporate a new point z_{n+1} on the same line into a set of n collinear points, find the nonzero coefficients d_1, d_2, d_{n+1} that ensure the collinearity of z_{n+1} with z_1 and z_2. Adding the equation $d_1 z_1 + d_2 z_2 + d_{n+1} z_{n+1} = 0$ to $\sum_{k=1}^{n} c_k z_k = 0$ creates a proper linear-dependency for the set of $n+1$ points. (**Note**: To ensure that the updated coefficients $c_1 + d_1$ and $c_2 + d_2$ are nonzero, one can always scale the set $\{d_1, d_2, d_{n+1}\}$ by a suitable constant such that the modified coefficients of z_1 and z_2 remain nonzero.)

With regard to the angle bisectors at the vertices B and C, we look for the scale factors s_1 and s_2 that yield the crossing point of these bisectors at $s_1 e^{\mathrm{i}\varphi_C/2} = B + s_2 e^{-\mathrm{i}\varphi_B/2}$. The real and imaginary parts of this equation yield

$$s_1 \cos(\tfrac{1}{2}\varphi_C) = a + b + s_2 \cos(\tfrac{1}{2}\varphi_B),$$

$$s_1 \sin(\tfrac{1}{2}\varphi_C) = -s_2 \sin(\tfrac{1}{2}\varphi_B).$$

Dividing the above equations, we find $\tan(\tfrac{1}{2}\varphi_C) = -s_2 \sin(\tfrac{1}{2}\varphi_B)/[a + b + s_2 \cos(\tfrac{1}{2}\varphi_B)]$, which yields the value of s_2 (and subsequently of s_1), as follows:

$$s_2 = -\frac{(a+b)\tan(\tfrac{1}{2}\varphi_C)}{\sin(\tfrac{1}{2}\varphi_B) + \cos(\tfrac{1}{2}\varphi_B)\tan(\tfrac{1}{2}\varphi_C)} = -\frac{(a+b)\sin(\tfrac{1}{2}\varphi_C)}{\sin[\tfrac{1}{2}(\varphi_B + \varphi_C)]}, \qquad s_1 = \frac{(a+b)\sin(\tfrac{1}{2}\varphi_B)}{\sin[\tfrac{1}{2}(\varphi_B + \varphi_C)]}.$$

Identical reasoning leads to a similar expression for the vertex A, with φ_B replacing $\tfrac{1}{2}\varphi_B$, and φ_C replacing $\tfrac{1}{2}\varphi_C$. Upon connecting the crossing point of the B and C bisectors to the vertex A, we find the following complex number (which must end up representing the bisector of A):

$$\begin{aligned}
&\frac{(a+b)\sin(\tfrac{1}{2}\varphi_B)}{\sin[\tfrac{1}{2}(\varphi_B+\varphi_C)]}e^{\mathrm{i}\varphi_C/2}-\frac{(a+b)\sin(\varphi_B)}{\sin(\varphi_B+\varphi_C)}e^{\mathrm{i}\varphi_C} \quad \leftarrow \boxed{\text{polar representation of vertex } A}\\
&\qquad=\frac{2(a+b)\sin(\tfrac{1}{2}\varphi_B)}{\sin(\varphi_B+\varphi_C)}e^{\mathrm{i}\varphi_C/2}\left\{\cos[\tfrac{1}{2}(\varphi_B+\varphi_C)]-\cos(\tfrac{1}{2}\varphi_B)\ e^{\mathrm{i}\varphi_C/2}\right\}\\
&\qquad=\frac{2(a+b)\sin(\tfrac{1}{2}\varphi_B)}{\sin(\varphi_B+\varphi_C)}e^{\mathrm{i}\varphi_C/2}[-\sin(\tfrac{1}{2}\varphi_B)\sin(\tfrac{1}{2}\varphi_C)-\mathrm{i}\cos(\tfrac{1}{2}\varphi_B)\sin(\tfrac{1}{2}\varphi_C)]\\
&\qquad=\frac{2(a+b)\sin(\tfrac{1}{2}\varphi_B)\sin(\tfrac{1}{2}\varphi_C)}{\sin(\varphi_B+\varphi_C)}e^{\mathrm{i}\varphi_C/2}e^{-\mathrm{i}\pi/2}[\cos(\tfrac{1}{2}\varphi_B)-\mathrm{i}\sin(\tfrac{1}{2}\varphi_B)]\\
&\qquad=\frac{2(a+b)\sin(\tfrac{1}{2}\varphi_B)\sin(\tfrac{1}{2}\varphi_C)}{\sin(\varphi_C+\varphi_B)}e^{-\mathrm{i}(\pi+\varphi_B-\varphi_C)/2}.
\end{aligned}$$

The phase of the above complex number is indeed the polar angle of the bisector of A, namely, $\varphi=-[\varphi_B+\tfrac{1}{2}(\pi-\varphi_B-\varphi_C)]$; the magnitude is the distance from the vertex A to the crossing point of the three angle bisectors.