Problem 2) a) The characteristic equation of the Fibonacci sequence is obtained by substituting the trial solution $f(n) = \zeta^n$ into the recurrence relation, yielding

$$\zeta^{n+2} = \zeta^n + \zeta^{n+1} \quad \rightarrow \quad \zeta^2 = 1 + \zeta \quad \rightarrow \quad \zeta^2 - \zeta - 1 = 0 \quad \rightarrow \quad \zeta_\pm = ½(1 \pm \sqrt{5}).$$

The resulting general solution is $f(n) = A\zeta_+^n + B\zeta_-^n$. Upon enforcing the initial conditions we obtain

$$f(0) = A + B = 0 \quad \rightarrow \quad B = -A.$$

$$f(1) = A\zeta_+ + B\zeta_- = 1 \quad \rightarrow \quad ½A(1 + \sqrt{5}) - ½A(1 - \sqrt{5}) = 1 \quad \rightarrow \quad A = 1/\sqrt{5}.$$

Consequently,

$$f(n) = \frac{[½(1+\sqrt{5})]^n - [½(1-\sqrt{5})]^n}{\sqrt{5}}.$$ ← Binet's formula

b) Summing the recurrence relation from $n = 0$ to N, we arrive at

$$\sum_{n=0}^{N} f(n+2) = \sum_{n=0}^{N} f(n) + \sum_{n=0}^{N} f(n+1)$$

$$\rightarrow \quad \sum_{m=2}^{N+2} f(m) = F(N) + \sum_{m=1}^{N+1} f(m)$$

$$\rightarrow \quad \cancel{F(N)} + \cancel{f(N+1)} + f(N+2) - \cancel{f(0)} - f(1) = \cancel{F(N)} + [F(N) + \cancel{f(N+1)} - \cancel{f(0)}]$$

$$\rightarrow \quad F(N) = f(N+2) - 1.$$

Alternatively, invoking the geometric series formula, we will have

$$F(N) = \sum_{n=0}^{N} f(n) = \frac{1}{\sqrt{5}}\sum_{n=0}^{N}[½(1+\sqrt{5})]^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{N}[½(1-\sqrt{5})]^n$$

$$= \frac{1}{\sqrt{5}}\left\{\frac{[½(1+\sqrt{5})]^{N+1} - 1}{½(1+\sqrt{5}) - 1} - \frac{[½(1-\sqrt{5})]^{N+1} - 1}{½(1-\sqrt{5}) - 1}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\frac{[½(1+\sqrt{5})]^{N+1} - 1}{-½(1-\sqrt{5})} - \frac{[½(1-\sqrt{5})]^{N+1} - 1}{-½(1+\sqrt{5})}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\frac{-[½(1+\sqrt{5})]^{N+2} + ½(1+\sqrt{5}) + [½(1-\sqrt{5})]^{N+2} - ½(1-\sqrt{5})}{¼(1-5)}\right\}$$

$$= \frac{1}{\sqrt{5}}\{[½(1+\sqrt{5})]^{N+2} - [½(1-\sqrt{5})]^{N+2} - \sqrt{5}\}$$

$$= f(N+2) - 1.$$